HL Paper 3

a. Find $\lim\limits_{x\to 0}\frac{\tan x}{x+x^2}$; [4]

b. Find $\lim\limits_{x\to 1}\frac{1-x^2+2x^2\ln x}{1-\sin\frac{\pi x}{2}}$. [7]

Markscheme

a. $\lim\limits_{x\to 0}\frac{\tan x}{x+x^2}=\lim\limits_{x\to 0}\frac{\sec^2 x}{1+2x}$ ***M1A1A1***

$\lim\limits_{x\to 0}\frac{\tan x}{x+x^2}=\frac{1}{1}=1$ ***A1***

[4 marks]

b. $\lim\limits_{x\to 1}\frac{1-x^2+2x^2\ln x}{1-\sin\frac{\pi x}{2}}=\lim\limits_{x\to 1}\frac{-2x+2x+4x\ln x}{-\frac{\pi}{2}\cos\frac{\pi x}{2}}$ ***M1A1A1***

$=\lim\limits_{x\to 1}\frac{4+4\ln x}{\frac{\pi^2}{4}\sin\frac{\pi x}{2}}$ ***M1A1A1***

$\lim\limits_{x\to 1}\frac{1-x^2+2x^2\ln x}{1-\sin\frac{\pi x}{2}}=\frac{4}{\frac{\pi^2}{4}}=\frac{16}{\pi^2}$ ***A1***

[7 marks]

Examiners report

a. This question was accessible to the vast majority of candidates, who recognised that L'Hopital's rule was required. A few of the weaker candidates did not realise that it needed to be applied twice in part (b). Many fully correct solutions were seen.

b. This question was accessible to the vast majority of candidates, who recognised that L'Hopital's rule was required. A few of the weaker candidates did not realise that it needed to be applied twice in part (b). Many fully correct solutions were seen.

Consider the differential equation

$\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^x+y\tan x$, given that $y = 1$ when $x = 0$.

The domain of the function y is $\left[0,\frac{\pi}{2}\right[$.

a. By finding the values of successive derivatives when $x = 0$, find the Maclaurin series for y as far as the term in x^3 . [6]

b. (i) Differentiate the function $\mathrm{e}^x(\sin x+\cos x)$ and hence show that [9]

$$\int \mathrm{e}^x\cos x\mathrm{d}x=\frac{1}{2}\mathrm{e}^x(\sin x+\cos x)+c.$$

(ii) Find an integrating factor for the differential equation and hence find the solution in the form $y=f(x)$.

Markscheme

a. we note that $y(0) = 1$ and $y'(0) = 2$ ***A1***

$y'' = 2\mathrm{e}^x + y' \tan x + y\sec^2 x$ ***M1***

$y''(0) = 3$ ***A1***

$y''' = 2\mathrm{e}^x + y'' \tan x + 2y'\sec^2 x + 2y\sec^2 x \tan x$ ***M1***

$y'''(0) = 6$ ***A1***

the maclaurin series solution is therefore

$y = 1 + 2x + \frac{3x^2}{2} + x^3 + \ldots$ ***A1***

[6 marks]

b. (i) $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x(\sin x + \cos x)) = \mathrm{e}^x(\sin x + \cos x) + \mathrm{e}^x(\cos x - \sin x)$ ***M1***

$= 2\mathrm{e}^x \cos x$ ***A1***

it follows that

$\int \mathrm{e}^x \cos x \mathrm{d}x = \frac{1}{2}\mathrm{e}^x(\sin x + \cos x) + c$ ***AG***

(ii) the differential equation can be written as

$\frac{\mathrm{d}y}{\mathrm{d}x} - y \tan x = 2\mathrm{e}^x$ ***M1***

$\text{IF} = \mathrm{e}^{\int -\tan x \mathrm{d}x} = \mathrm{e}^{\ln \cos x} = \cos x$ ***M1A1***

$\cos x \frac{\mathrm{d}y}{\mathrm{d}x} - y \sin x = 2\mathrm{e}^x \cos x$ ***M1***

integrating,

$y \cos x = \mathrm{e}^x(\sin x + \cos x) + C$ ***A1***

$y = 1$ when $x = 0$ gives $C = 0$ ***M1***

therefore

$y = \mathrm{e}^x(1 + \tan x)$ ***A1***

[9 marks]

Examiners report

a. [N/A]

b. [N/A]

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a. (i) Show that $\int_1^\infty \frac{1}{x(x+p)} \mathrm{d}x,\ p \neq 0$ is convergent if $p > -1$ and find its value in terms of p. [8]

(ii) Hence show that the following series is convergent.

$$\frac{1}{1 \times 0.5} + \frac{1}{2 \times 1.5} + \frac{1}{3 \times 2.5} + \ldots$$

b. Determine, for each of the following series, whether it is convergent or divergent. [11]

(i) $\sum\limits_{n=1}^{\infty} \sin\left(\frac{1}{n(n+3)}\right)$

(ii) $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{6}} + \sqrt{\frac{1}{12}} + \sqrt{\frac{1}{20}} + \ldots$

Markscheme

a. (i) the integrand is non-singular on the domain if $p > -1$ with the latter assumed, consider

$\int_1^R \frac{1}{x(x+p)}\mathrm{d}x = \frac{1}{p}\int_1^R \frac{1}{x} - \frac{1}{x+p}\mathrm{d}x$ ***M1A1***

$= \frac{1}{p}\left[\ln\left(\frac{x}{x+p}\right)\right]_1^R,\ p \neq 0$ ***A1***

this evaluates to

$\frac{1}{p}\left(\ln\frac{R}{R+p} - \ln\frac{1}{1+p}\right),\ p \neq 0$ ***M1***

$\to \frac{1}{p}\ln(1+p)$ ***A1***

because $\frac{R}{R+p} \to 1$ as $R \to \infty$ ***R1***

hence the integral is convergent ***AG***

(ii) the given series is $\sum\limits_{n=1}^{\infty} f(n),\ f(n) = \frac{1}{n(n-0.5)}$ ***M1***

the integral test and $p = -0.5$ in (i) establishes the convergence of the series ***R1***

[8 marks]

b. (i) as we have a series of positive terms we can apply the comparison test, limit form

comparing with $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ ***M1***

$\lim\limits_{n\to\infty} \frac{\sin\left(\frac{1}{n(n+3)}\right)}{\frac{1}{n^2}} = 1$ ***M1A1***

as $\sin\theta \approx \theta$ for small θ ***R1***

and $\frac{n^2}{n(n+3)} \to 1$ ***R1***

(so as the limit (of 1) is finite and non-zero, both series exhibit the same behavior)

$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, so this series converges ***R1***

(ii) the general term is

$\sqrt{\frac{1}{n(n+1)}}$ ***A1***

$\sqrt{\frac{1}{n(n+1)}} > \sqrt{\frac{1}{(n+1)(n+1)}}$ ***M1***

$\sqrt{\frac{1}{(n+1)(n+1)}} = \frac{1}{n+1}$ ***A1***

the harmonic series diverges ***R1***

so by the comparison test so does the given series ***R1***

[11 marks]

Examiners report

a. Part(a)(i) caused problems for some candidates who failed to realize that the integral can only be tackled by the use of partial fractions. Even then, the improper integral only exists as a limit – too many candidates ignored or skated over this important point. Candidates must realize that in this type of question, rigour is important, and full marks will only be awarded for a full and clearly explained argument. This applies as well to part(b), where it was also noted that some candidates were confusing the convergence of the terms of a series to zero with convergence of the series itself.

b. Part(a)(i) caused problems for some candidates who failed to realize that the integral can only be tackled by the use of partial fractions. Even then, the improper integral only exists as a limit – too many candidates ignored or skated over this important point. Candidates must realize that in this type of question, rigour is important, and full marks will only be awarded for a full and clearly explained argument. This applies as well to part(b), where it was also noted that some candidates were confusing the convergence of the terms of a series to zero with convergence of the series itself.

The function *f* is defined by

$$f(x) = \ln\left(\frac{1}{1-x}\right).$$

(a) Write down the value of the constant term in the Maclaurin series for $f(x)$.

(b) Find the first three derivatives of $f(x)$ and hence show that the Maclaurin series for $f(x)$ up to and including the x^3 term is $x + \frac{x^2}{2} + \frac{x^3}{3}$.

(c) Use this series to find an approximate value for ln 2 .

(d) Use the Lagrange form of the remainder to find an upper bound for the error in this approximation.

(e) How good is this upper bound as an estimate for the actual error?

Markscheme

(a) Constant term = 0 ***A1***

[1 mark]

(b) $f'(x) = \frac{1}{1-x}$ ***A1***

$f''(x) = \frac{1}{(1-x)^2}$ ***A1***

$f'''(x) = \frac{2}{(1-x)^3}$ ***A1***

$f'(0) = 1;\ f''(0) = 1;\ f'''(0) = 2$ ***A1***

Note: Allow ***FT*** on their derivatives.

$f(x) = 0 + \frac{1 \times x}{1!} + \frac{1 \times x^2}{2!} + \frac{2 \times x^3}{3!} + \ldots$ ***M1A1***

$= x + \frac{x^2}{2} + \frac{x^3}{3}$ ***AG***

[6 marks]

(c) $\frac{1}{1-x} = 2 \Rightarrow x = \frac{1}{2}$ ***(A1)***

$\ln 2 \approx \frac{1}{2} + \frac{1}{8} + \frac{1}{24}$ ***M1***

$= \frac{2}{3}$ (0.667) ***A1***

[3 marks]

(d) Lagrange error $= \frac{f^{(n+1)}(c)}{(n+1)!} \times \left(\frac{1}{2}\right)^{n+1}$ ***(M1)***

$= \frac{6}{(1-c)^4} \times \frac{1}{24} \times \left(\frac{1}{2}\right)^4$ ***A1***

$< \frac{6}{\left(1-\frac{1}{2}\right)^4} \times \frac{1}{24} \times \frac{1}{16}$ ***A2***

giving an upper bound of 0.25. ***A1***

[5 marks]

(e) Actual error $= \ln 2 - \frac{2}{3} = 0.0265$ ***A1***

The upper bound calculated is much larger that the actual error therefore cannot be considered a good estimate. ***R1***

[2 marks]

Total [17 marks]

Examiners report

In (a), some candidates appeared not to understand the term ‘constant term’. In (b), many candidates found the differentiation beyond them with only a handful realising that the best way to proceed was to rewrite the function as $f(x) = -\ln(1-x)$. In (d), many candidates were unable to use the Lagrange formula for the upper bound so that (e) became inaccessible.

The function *f* is defined on the domain $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ by $f(x) = \ln(1 + \sin x)$.

a. Show that $f''(x) = -\frac{1}{(1+\sin x)}$. [4]

b. (i) Find the Maclaurin series for $f(x)$ up to and including the term in x^4 . [7]

(ii) Explain briefly why your result shows that *f* is neither an even function nor an odd function.

c. Determine the value of $\lim\limits_{x \to 0} \frac{\ln(1+\sin x)-x}{x^2}$. [3]

Markscheme

a. $f'(x) = \frac{\cos x}{1+\sin x}$ ***A1***

$f''(x) = \frac{-\sin x(1+\sin x)-\cos x\cos x}{(1+\sin x)^2}$ ***M1A1***

$= \frac{-\sin x-(\sin^2 x+\cos^2 x)}{(1+\sin x)^2}$ ***A1***

$= -\frac{1}{1+\sin x}$ ***AG***

[4 marks]

b. (i) $f'''(x) = \frac{\cos x}{(1+\sin x)^2}$ ***A1***

$f^{(4)}(x) = \frac{-\sin x(1+\sin x)^2-2(1+\sin x)\cos^2 x}{(1+\sin x)^4}$ ***M1A1***

$f(0) = 0,\ f'(0) = 1,\ f''(0) = -1$ ***M1***

$f'''(0) = 1,\ f^{(4)}(0) = -2$ ***A1***

$f(x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$ ***A1***

(ii) the series contains even and odd powers of *x* ***R1***

[7 marks]

c. $\lim\limits_{x \to 0} \frac{\ln(1+\sin x) - x}{x^2} = \lim\limits_{x \to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots - x}{x^2}$ ***M1***

$= \lim\limits_{x \to 0} \frac{\frac{-1}{2} + \frac{x}{6} + \ldots}{1}$ ***(A1)***

$= -\frac{1}{2}$ ***A1***

Note: Use of l'Hopital's Rule is also acceptable.

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The exponential series is given by $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$.

a. Find the set of values of *x* for which the series is convergent. [4]

b. (i) Show, by comparison with an appropriate geometric series, that [6]

$$e^x - 1 < \frac{2x}{2-x}, \text{ for } 0 < x < 2.$$

(ii) Hence show that $e < \left(\frac{2n+1}{2n-1}\right)^n$, for $n \in \mathbb{Z}^+$.

c. (i) Write down the first three terms of the Maclaurin series for $1 - e^{-x}$ and explain why you are able to state that [4]

$$1 - e^{-x} > x - \frac{x^2}{2}, \text{ for } 0 < x < 2.$$

(ii) Deduce that $e > \left(\frac{2n^2}{2n^2-2n+1}\right)^n$, for $n \in \mathbb{Z}^+$.

d. Letting *n* = 1000, use the results in parts (b) and (c) to calculate the value of e correct to as many decimal places as possible. [2]

Markscheme

a. using a ratio test,

$\left|\frac{T_{n+1}}{T_n}\right| = \left|\frac{x^{n+1}}{(n+1)!}\right| \times \left|\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$ ***M1A1***

Note: Condone omission of modulus signs.

$\to 0$ as $n \to \infty$ for all values of *x* ***R1*** the series is therefore convergent for $x \in \mathbb{R}$ ***A1***

[4 marks]

b. (i) $\mathrm{e}^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{2\times 3} + \ldots$ ***M1***

$< x + \frac{x^2}{2} + \frac{x^3}{2\times 2} + \ldots \quad (\text{for } x > 0)$ ***A1***

$= \frac{x}{1-\frac{x}{2}} \quad (\text{for } x < 2)$ ***A1***

$= \frac{2x}{2-x} \quad (\text{for } 0 < x < 2)$ ***AG***

(ii) $\mathrm{e}^x < 1 + \frac{2x}{2-x} = \frac{2+x}{2-x}$ ***A1***

$\mathrm{e} < \left(\frac{2+x}{2-x}\right)^{\frac{1}{x}}$ ***A1***

replacing *x* by $\frac{1}{n}$ (and noting that the result is true for $n > \frac{1}{2}$ and therefore $\mathbb{Z}^+$) ***M1***

$\mathrm{e} < \left(\frac{2n+1}{2n-1}\right)^n$ ***AG***

[6 marks]

c. (i) $1 - \mathrm{e}^{-x} = x - \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ ***A1***

for $0 < x < 2$, the series is alternating with decreasing terms so that the sum is greater than the sum of an even number of terms ***R1***

therefore

$1 - \mathrm{e}^{-x} > x - \frac{x^2}{2}$ ***AG***

(ii) $\mathrm{e}^{-x} < 1 - x + \frac{x^2}{2}$

$\mathrm{e}^x > \frac{1}{\left(1-x+\frac{x^2}{2}\right)}$ ***M1***

$\mathrm{e} > \left(\frac{2}{2-2x+x^2}\right)^{\frac{1}{x}}$ ***A1***

replacing *x* by $\frac{1}{n}$ (and noting that the result is true for $n > \frac{1}{2}$ and therefore $\mathbb{Z}^+$)

$\mathrm{e} > \left(\frac{2n^2}{2n^2-2n+1}\right)^n$ ***AG***

[4 marks]

d. from (b) and (c), $\mathrm{e} < 2.718282\ldots$ and $\mathrm{e} > 2.718281\ldots$ ***A1***

we conclude that e = 2.71828 correct to 5 decimal places ***A1***

[2 marks]

Examiners report

a. Solutions to (a) were generally good although some candidates failed to reach the correct conclusion from correct application of the ratio test.

Solutions to (b) and (c), however, were generally disappointing with many candidates unable to make use of the signposting in the question.

Candidates who were unable to solve (b) and (c) often picked up marks in (d).

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Determine whether or not the following series converge.

(a) $\sum\limits_{n=0}^{\infty}\left(\sin\frac{n\pi}{2}-\sin\frac{(n+1)\pi}{2}\right)$

(b) $\sum\limits_{n=1}^{\infty}\frac{\mathrm{e}^n-1}{\pi^n}$

(c) $\sum\limits_{n=2}^{\infty}\frac{\sqrt{n+1}}{n(n-1)}$

Markscheme

(a) $\sum\limits_{n=0}^{\infty}\left(\sin\frac{n\pi}{2}-\sin\frac{(n+1)\pi}{2}\right)$

$=\left(\sin 0-\sin\frac{\pi}{2}\right)+\left(\sin\frac{\pi}{2}-\sin\pi\right)+\left(\sin\pi-\sin\frac{3\pi}{2}\right)+\left(\sin\frac{3\pi}{2}-\sin 2\pi\right)+\ldots$ ***(M1)***

the n^{th} term is ± 1 for all n, *i.e.* the n^{th} term does not tend to 0 ***A1***

hence the series does not converge ***A1***

[3 marks]

(b) **EITHER**

using the ratio test ***(M1)***

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\left(\frac{\mathrm{e}^{n+1}}{\pi^{n+1}}\right)\left(\frac{\pi^n}{\mathrm{e}^n-1}\right)$ ***M1A1***

$\lim\limits_{n\to\infty}\left(\frac{\mathrm{e}^{n+1}-1}{\mathrm{e}^n-1}\right)\left(\frac{\pi^n}{\pi^{n+1}}\right)=\frac{\mathrm{e}}{\pi}\quad(\approx 0.865)$ ***M1A1***

$\frac{\mathrm{e}}{\pi}<1$, hence the series converges ***R1A1***

OR

$\sum\limits_{n=1}^{\infty}\frac{\mathrm{e}^n-1}{\pi^n}=\sum\limits_{n=1}^{\infty}\left(\frac{\mathrm{e}}{\pi}\right)^n-\left(\frac{1}{\pi}\right)^n=\sum\limits_{n=1}^{\infty}\left(\frac{\mathrm{e}}{\pi}\right)^n-\sum\limits_{n=1}^{\infty}\left(\frac{1}{\pi}\right)^n$ ***M1A1***

the series is the difference of two geometric series, with $r=\frac{\mathrm{e}}{\pi}\quad(\approx 0.865)$ ***M1A1***

and $\frac{1}{\pi}\quad(\approx 0.318)$ ***A1***

for both $|r|<1$, hence the series converges ***R1A1***

OR

$\forall n,\ 0<\frac{\mathrm{e}^n-1}{\pi^n}<\frac{\mathrm{e}^n}{\pi^n}$ ***(M1)A1A1*** the series $\frac{e^n}{\pi^n}$ converges since it is a geometric series such that $|r| < 1$ ***A1R1***

therefore, by the comparison test, $\frac{e^n - 1}{\pi^n}$ converges ***R1A1***

[7 marks]

(c) by limit comparison test with $\frac{\sqrt{n}}{n^2}$, ***(M1)***

$\lim_{n \to \infty} \left(\frac{\frac{\sqrt{n+1}}{n(n-1)}}{\frac{\sqrt{n}}{n^2}} \right) = \lim_{n \to \infty} \left(\frac{\sqrt{n+1}}{n(n-1)} \times \frac{n^2}{\sqrt{n}} \right) = \lim_{n \to \infty} \frac{n}{n-1} \sqrt{\frac{n+1}{n}} = 1$ ***M1A1***

hence both series converge or both diverge ***R1***

by the p-test $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = n^{\frac{-3}{2}}$ converges, hence both converge ***R1A1***

[6 marks]

Total [16 marks]

Examiners report

This was the least successfully answered question on the paper. Candidates often did not know which convergence test to use; hence very few full successful solutions were seen. The communication of the method used was often quite poor.

a) Many candidates failed to see that this is a telescoping series. If this was recognized then the question was fairly straightforward. Often candidates unsuccessfully attempted to apply the standard convergence tests.
b) Many candidates used the ratio test, but some had difficulty in simplifying the expression. Others recognized that the series is the difference of two geometric series, and although the algebraic work was done correctly, some failed to communicate the conclusion that since the absolute value of the ratios are less than 1, hence the series converges. Some candidates successfully used the comparison test.
c) Although the limit comparison test was attempted by most candidates, it often failed through an inappropriate selection of a series.

Find $\lim_{x \to 0} \left(\frac{1 - \cos x^6}{x^{12}} \right)$.

Markscheme

METHOD 1

$f(0) = \frac{0}{0}$, hence using l'Hôpital's Rule, ***(M1)***

$g(x) = 1 - \cos(x^6),\ h(x) = x^{12};\ \frac{g'(x)}{h'(x)} = \frac{6x^5 \sin(x^6)}{12x^{11}} = \frac{\sin(x^6)}{2x^6}$ ***A1A1***

EITHER

$\frac{g'(0)}{h'(0)} = \frac{0}{0}$, using l'Hôpital's Rule again, ***(M1)***

$\frac{g''(x)}{h''(x)} = \frac{6x^5 \cos(x^6)}{12x^5} = \frac{\cos(x^6)}{2}$ ***A1A1***

$\frac{g''(0)}{h''(0)} = \frac{1}{2}$, hence the limit is $\frac{1}{2}$ ***A1***

OR

So $\lim_{x \to 0} \frac{1 - \cos x^6}{x^{12}} = \lim_{x \to 0} \frac{\sin x^6}{2x^6}$ ***A1***

$= \frac{1}{2} \lim_{x \to 0} \frac{\sin x^6}{2x^6}$ ***A1***

$= \frac{1}{2}$ since $\lim_{x \to 0} \frac{\sin x^6}{2x^6} = 1$ ***A1 (R1)***

METHOD 2

substituting x^6 for x in the expansion $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} \ldots$ ***(M1)***

$\frac{1 - \cos x^6}{x^{12}} = \frac{1 - \left(1 - \frac{x^{12}}{2} + \frac{x^{24}}{24}\right) \ldots}{x^{12}}$ ***M1A1***

$= \frac{1}{2} - \frac{x^{12}}{24} + \ldots$ ***A1A1***

$\lim_{x \to 0} \frac{1 - \cos x^6}{x^{12}} = \frac{1}{2}$ ***M1A1***

Note: Accept solutions using Maclaurin expansions.

[7 marks]

Examiners report

Surprisingly, some weaker candidates were more successful in answering this question than stronger candidates. If candidates failed to simplify the expression after the first application of L'Hôpital's rule, they generally were not successful in correctly differentiating the expression a 2^{nd} time, hence could not achieve the final three A marks.

(a) Given that $y = \ln \cos x$, show that the first two non-zero terms of the Maclaurin series for y are $-\frac{x^2}{2} - \frac{x^4}{12}$.

(b) Use this series to find an approximation in terms of π for $\ln 2$.

Markscheme

(a) $f(x) = \ln \cos x$

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$ ***M1A1***

$f''(x) = -\sec^2 x$ ***M1***

$f'''(x) = -2 \sec x \sec x \tan x$ ***A1***

$f^{iv}(x) = -2\sec^2 x(\sec^2 x) - 2 \tan x(2\sec^2 x \tan x)$

$= -2\sec^4 x - 4\sec^2 x \tan^2 x$ ***A1***

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$

$f(0) = 0,$ ***M1***

$f'(0) = 0,$

$f''(0) = -1,$

$f'''(0) = 0,$

$f^{iv}(0) = -2,$ ***A1***

Notes: Award the ***A1*** if all the substitutions are correct.

Allow ***FT*** from their derivatives.

$\ln(\cos x) \approx -\frac{x^2}{2!} - \frac{2x^4}{4!}$ ***A1***

$= -\frac{x^2}{2} - \frac{x^4}{12}$ ***AG***

[8 marks]

(b) Some consideration of the manipulation of ln 2 ***(M1)***

Attempt to find an angle ***(M1)***

EITHER

Taking $x = \frac{\pi}{3}$ ***A1***

$\ln \frac{1}{2} \approx -\frac{\left(\frac{\pi}{3}\right)^2}{2!} - \frac{2\left(\frac{\pi}{3}\right)^4}{4!}$ ***A1***

$-\ln 2 \approx -\frac{\frac{\pi^2}{9}}{2!} - \frac{2\frac{\pi^4}{81}}{4!}$ ***A1***

$\ln 2 \approx \frac{\pi^2}{18} + \frac{\pi^4}{972} = \frac{\pi^2}{9}\left(\frac{1}{2} + \frac{\pi^2}{108}\right)$ ***A1***

OR

Taking $x = \frac{\pi}{4}$ ***A1***

$\ln \frac{1}{\sqrt{2}} \approx -\frac{\left(\frac{\pi}{4}\right)^2}{2!} - \frac{2\left(\frac{\pi}{4}\right)^4}{4!}$ ***A1***

$-\frac{1}{2}\ln 2 \approx -\frac{\frac{\pi^2}{16}}{2!} - \frac{2\frac{\pi^4}{256}}{4!}$ ***A1***

$\ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536} = \frac{\pi^2}{8}\left(\frac{1}{2} + \frac{\pi^2}{192}\right)$ ***A1***

[6 marks]

Total [14 marks]

Examiners report

Some candidates had difficulty organizing the derivatives but most were successful in getting the series. Using the series to find the approximation for $\ln 2$ in terms of π was another story and it was rare to see a good solution.

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$ where $y = 1$ when $x = 0$.

a. Use Euler's method with step length 0.1 to find an approximate value of y when $x = 0.4$. [7]

b. Write down, giving a reason, whether your approximate value for y is greater than or less than the actual value of y . [1]

Markscheme

a. use of $y \to y + h\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

x	y	$\frac{dy}{dx}$	$h\frac{dy}{dx}$	
0	1	1	0.1	***A1***
0.1	1.1	1.22	0.122	***A1***
0.2	1.222	1.533284	0.1533284	***A1***
0.3	1.3753284	1.981528208	0.1981528208	***A1***
0.4	1.573481221			***(A1)***

approximate value of $y = 1.57$ ***A1***

Note: Accept values in the tables correct to 3 significant figures.

[7 marks]

b. the approximate value is less than the actual value because it is assumed that $\frac{dy}{dx}$ remains constant throughout each interval whereas it is actually an increasing function ***R1***

[1 mark]

Examiners report

a. Most candidates were familiar with Euler's method. The most common way of losing marks was either to round intermediate answers to insufficient accuracy or simply to make an arithmetic error. Many candidates were given an accuracy penalty for not rounding their answer to three significant figures. Few candidates were able to answer (b) correctly with most believing incorrectly that the step length was a relevant factor.

b. Most candidates were familiar with Euler's method. The most common way of losing marks was either to round intermediate answers to insufficient accuracy or simply to make an arithmetic error. Many candidates were given an accuracy penalty for not rounding their answer to three significant figures. Few candidates were able to answer (b) correctly with most believing incorrectly that the step length was a relevant factor.

Consider the function $f(x) = \sin(p \arcsin x),\ -1 < x < 1$ and $p \in \mathbb{R}$.

The function f and its derivatives satisfy

$$(1 - x^2)f^{(n+2)}(x) - (2n + 1)xf^{(n+1)}(x) + (p^2 - n^2)f^{(n)}(x) = 0,\ n \in \mathbb{N}$$

where $f^{(n)}(x)$ denotes the n th derivative of $f(x)$ and $f^{(0)}(x)$ is $f(x)$.

a. Show that $f'(0) = p$. [2]

b. Show that $f^{(n+2)}(0) = (n^2 - p^2)f^{(n)}(0)$. [1]

c. For $p \in \mathbb{R}\backslash\{\pm 1,\ \pm 3\}$, show that the Maclaurin series for $f(x)$, up to and including the x^5 term, is [4]

$$px + \frac{p(1-p^2)}{3!}x^3 + \frac{p(9-p^2)(1-p^2)}{5!}x^5.$$

d. Hence or otherwise, find $\lim\limits_{x \to 0} \frac{\sin(p \arcsin x)}{x}$. [2]

e. If p is an odd integer, prove that the Maclaurin series for $f(x)$ is a polynomial of degree p. [4]

Markscheme

a. $f'(x) = \frac{p\cos(p\arcsin x)}{\sqrt{1-x^2}}$ ***(M1)A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f'(0) = p$ ***AG***

[2 marks]

b. **EITHER**

$f^{(n+2)}(0) + (p^2 - n^2)f^{(n)}(0) = 0$ ***A1***

OR

for *eg*, $(1-x^2)f^{(n+2)}(x) = (2n+1)xf^{(n+1)}(x) - (p^2 - n^2)f^{(n)}(x)$ ***A1***

Note: Award ***A1*** for *eg*, $(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) = -(p^2-n^2)f^{(n)}(x)$.

THEN

$f^{(n+2)}(0) = (n^2 - p^2)f^{(n)}(0)$ ***AG***

[1 mark]

c. considering f and its derivatives at $x = 0$ ***(M1)***

$f(0) = 0$ and $f'(0) = p$ from (a) ***A1***

$f''(0) = 0,\ f^{(4)}(0) = 0$ ***A1***

$f^{(3)}(0) = (1-p^2)f^{(1)}(0) = (1-p^2)p,$

$f^{(5)}(0) = (9-p^2)f^{(3)}(0) = (9-p^2)(1-p^2)p$ ***A1***

Note: Only award the last ***A1*** if either $f^{(3)}(0) = (1-p^2)f^{(1)}(0)$ and $f^{(5)}(0) = (9-p^2)f^{(3)}(0)$ have been stated or the general Maclaurin series has been stated and used.

$px + \frac{p(1-p^2)}{3!}x^3 + \frac{p(9-p^2)(1-p^2)}{5!}x^5$ ***AG***

[4 marks]

d. **METHOD 1**

$\lim_{x\to 0}\frac{\sin(p\arcsin x)}{x} = \lim_{x\to 0}\frac{px+\frac{p(1-p^2)}{3!}x^3+\ldots}{3}$ ***M1***

$= p$ ***A1***

METHOD 2

by l'Hôpital's rule $\lim_{x\to 0}\frac{\sin(p\arcsin x)}{x} = \lim_{x\to 0}\frac{p\cos(p\arcsin x)}{\sqrt{1-x^2}}$ ***M1***

$= p$ ***A1***

[2 marks]

e. the coefficients of all even powers of x are zero ***A1***

the coefficient of x^p for (p odd) is non-zero (or equivalent *eg*,

the coefficients of all odd powers of x up to p are non-zero) ***A1***

$f^{(p+2)}(0) = (p^2 - p^2) f^{(p)}(0) = 0$ and so the coefficient of x^{p+2} is zero ***A1***

the coefficients of all odd powers of x greater than $p+2$ are zero (or equivalent) ***A1***

so the Maclaurin series for $f(x)$ is a polynomial of degree p ***AG***

[4 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

---

a. Given that $n > \ln n$ for $n > 0$, use the comparison test to show that the series $\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)}$ is divergent. [3]

b. Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{(3x)^n}{\ln(n+2)}$. [7]

Markscheme

a. **METHOD 1**

$\ln(n+2) < n+2$ ***(A1)***

$\Rightarrow \frac{1}{\ln(n+2)} > \frac{1}{n+2}$ (for $n \geqslant 0$) ***A1***

Note: Award A0 for statements such as $\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)} > \sum_{n=0}^{\infty} \frac{1}{n+2}$. However condone such a statement if the above ***A1*** has already been awarded.

$\sum_{n=0}^{\infty} \frac{1}{n+2}$ (is a harmonic series which) diverges ***R1***

Note: The ***R1*** is independent of the ***A1***s.

Award ***R0*** for statements such as "$\frac{1}{n+2}$ diverges".

so $\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)}$ diverges by the comparison test ***AG***

METHOD 2

$\frac{1}{\ln n} > \frac{1}{n}$ (for $n \geqslant 2$) ***A1***

Note: Award ***A0*** for statements such as $\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$. However condone such a statement if the above ***A1*** has already been awarded.

a correct statement linking n and $n+2$ *eg*,

$\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ or $\sum_{n=0}^{\infty} \frac{1}{n+2} = \sum_{n=2}^{\infty} \frac{1}{n}$ **A1**

Note: Award ***A0*** for $\sum_{n=0}^{\infty} \frac{1}{n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ (is a harmonic series which) diverges

(which implies that $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by the comparison test) ***R1***

Note: The ***R1*** is independent of the ***A1***s.

Award ***R0*** for statements such as $\sum_{n=0}^{\infty} \frac{1}{n}$ deiverges and "$\frac{1}{n}$ diverges".

Award ***A1A0R1*** for arguments based on $\sum_{n=1}^{\infty} \frac{1}{n}$.

so $\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)}$ diverges by the comparison test ***AG***

[3 marks]

b. applying the ratio test $\lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{\ln(n+3)} \times \frac{\ln(n+2)}{(3x)^n}\right|$ ***M1***

$= |3x|$ (as $\lim_{n\to\infty}\left|\frac{\ln(n+2)}{\ln(n+3)}\right| = 1$ ***A1***

Note: Condone the absence of limits and modulus signs.

Note: Award ***M1A0*** for $3x^n$. Subsequent marks can be awarded.

series converges for $-\frac{1}{3} < x < \frac{1}{3}$

considering $x = -\frac{1}{3}$ and $x = \frac{1}{3}$ ***M1***

Note: Award ***M1*** to candidates who consider one endpoint.

when $x = \frac{1}{3}$, series is $\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)}$ which is divergent (from (a)) ***A1***

Note: Award this ***A1*** if $\sum_{n=0}^{\infty} \frac{1}{\ln(n+2)}$ is not stated but reference to part (a) is.

when $x = -\frac{1}{3}$, series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{\ln(n+2)}$ ***A1***

$\sum_{n=0}^{\infty} \frac{(-1)^n}{\ln(n+2)}$ converges (conditionally) by the alternating series test ***R1***

(strictly alternating, $|u_n| > |u_{n+1}|$ for $n \geqslant 0$ and $\lim_{n\to\infty}(u_n) = 0$)

so the interval of convergence of *S* is $-\frac{1}{3} \leqslant x < \frac{1}{3}$ ***A1***

Note: The final ***A1*** is dependent on previous ***A1***s – *ie*, considering correct series when $x = -\frac{1}{3}$ and $x = \frac{1}{3}$ and on the final ***R1***.

Award as above to candidates who firstly consider $x = -\frac{1}{3}$ and then state conditional convergence implies divergence at $x = \frac{1}{3}$.

[7 marks]

Examiners report

a. [N/A]
b. [N/A]

Let the Maclaurin series for $\tan x$ be

$$\tan x = a_1 x + a_3 x^3 + a_5 x^5 + \ldots$$

where a_1, a_3 and a_5 are constants.

a.i. Find series for $\sec^2 x$, in terms of a_1, a_3 and a_5, up to and including the x^4 term [1]

by differentiating the above series for $\tan x$;

a.ii. Find series for $\sec^2 x$, in terms of a_1, a_3 and a_5, up to and including the x^4 term [2]

by using the relationship $\sec^2 x = 1 + \tan^2 x$.

b. Hence, by comparing your two series, determine the values of a_1, a_3 and a_5. [3]

Markscheme

a.i. $(\sec^2 x =)\ a_1 + 3a_3 x^2 + 5a_5 x^4 + \ldots$ ***A1***

[1 mark]

a.ii. $\sec^2 x = 1 + (a_1 x + a_3 x^3 + a_5 x^5 + \ldots)^2$

$= 1 + a_1^2 x^2 + 2a_1 a_3 x^4 + \ldots$ ***M1A1***

Note: Condone the presence of terms with powers greater than four.

[2 marks]

b. equating constant terms: $a_1 = 1$ ***A1***

equating x^2 terms: $3a_3 = a_1^2 = 1 \Rightarrow a_3 = \frac{1}{3}$ ***A1***

equating x^4 terms: $5a_5 = 2a_1 a_3 = \frac{2}{3} \Rightarrow a_5 = \frac{2}{15}$ ***A1***

[3 marks]

Examiners report

a.i. [N/A]
a.ii. [N/A]
b. [N/A]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} - xy$ where $y > 0$ and $y = 2$ when $x = 0$.

a. Show that putting $z = y^2$ transforms the differential equation into $\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 2x$. [4]

b. By solving this differential equation in z, obtain an expression for y in terms of x. [9]

Markscheme

a. **METHOD 1**

$z = y^2 \Rightarrow y = z^{1/2}$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2z^{1/2}}\frac{\mathrm{d}z}{\mathrm{d}x}$ ***M1A1***

substituting, $\frac{1}{2z^{1/2}}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x}{z^{1/2}} - xz^{1/2}$ ***M1A1***

$\frac{\mathrm{d}z}{\mathrm{d}x} + 2xz = 2x$ ***AG***

METHOD 2

$z = y^2$

$\frac{\mathrm{d}z}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1***

$\frac{\mathrm{d}z}{\mathrm{d}x} = 2x - 2xy^2$ ***M1A1***

$\frac{\mathrm{d}z}{\mathrm{d}x} = 2xz = 2x$ ***AG***

[4 marks]

b. **METHOD 1**

integrating factor $= \mathrm{e}^{\int 2x\mathrm{d}x} = \mathrm{e}^{x^2}$ ***(M1)A1***

$\mathrm{e}^{x^2}\frac{\mathrm{d}z}{\mathrm{d}x} + 2x\mathrm{e}^{x^2}z = 2x\mathrm{e}^{x^2}$ ***(M1)***

$z\mathrm{e}^{x^2} = \int 2x\mathrm{e}^{x^2}\mathrm{d}x$ ***A1***

$= \mathrm{e}^{x^2} + C$ ***A1***

substitute $y = 2$ therefore $z = 4$ when $x = 0$ ***(M1)***

$4 = 1 + C$

$C = 3$ ***(A1)***

the solution is $z = 1 + 3\mathrm{e}^{-x^2}$ ***(M1)***

Note: This line may be seen before determining the value of C.

so that $y = \sqrt{1 + 3\mathrm{e}^{-x^2}}$ ***A1***

METHOD 2

$\frac{\mathrm{d}z}{\mathrm{d}x} = 2x(1 - z)$

$\int \frac{1}{1-z}\mathrm{d}z = \int 2x\mathrm{d}x$ ***M1***

$-\ln(1 - z) = x^2 + C$ ***A1A1***

$1 - z = \mathrm{e}^{-x^2-c}$ (or $1 - z = B\mathrm{e}^{-x^2}$) ***M1A1***

solving for z ***(M1)***

$z = 1 + A\mathrm{e}^{-x^2}$

$z = 4$ when $x = 0$ ***(M1)***

so $A = 3$ ***(A1)***

the solution is $z = 1 + 3\mathrm{e}^{-x^2}$

so $y = \sqrt{1 + 3\mathrm{e}^{-x^2}}$ ***A1***

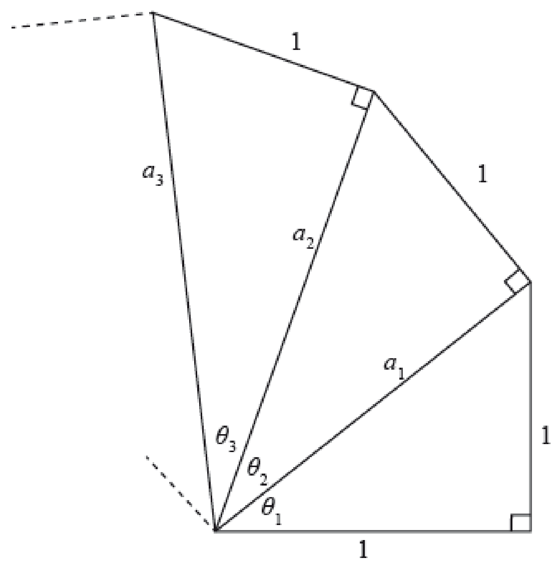
[9 marks]

Examiners report

a. Several misconceptions were identified that showed poor understanding of the chain rule. Although many candidates were successful in establishing the result the presentation of their work was far from what is expected in a show that question.

b. Part (b) was well attempted using both method 1 (integration factor) and 2 (separation of variables). The most common error was omission of the constant of integration or errors in finding its value. Candidates that used method 2 often had difficulties in integrating $\frac{1}{(1-z)}$ correctly and making z the subject often losing out on accuracy marks.

Consider the infinite spiral of right angle triangles as shown in the following diagram.



The $n\text{th}$ triangle in the spiral has central angle θ_n, hypotenuse of length a_n and opposite side of length 1, as shown in the diagram. The first right angle triangle is isosceles with the two equal sides being of length 1.

Consider the series $\sum\limits_{n=1}^{\infty} \theta_n$.

a. Using l'Hôpital's rule, find $\lim\limits_{x \to \infty} \left(\frac{\arcsin\left(\frac{1}{\sqrt{(x+1)}}\right)}{\frac{1}{\sqrt{x}}} \right)$. [6]

b. (i) Find a_1 and a_2 and hence write down an expression for a_n. [3]

(ii) Show that $\theta_n = \arcsin \frac{1}{\sqrt{(n+1)}}$.

c. Using a suitable test, determine whether this series converges or diverges. [6]

Markscheme

a. $\lim\limits_{x \to \infty} \left(\frac{\arcsin\left(\frac{1}{\sqrt{(x+1)}}\right)}{\frac{1}{\sqrt{x}}} \right)$ is of the form $\frac{0}{0}$

and so will equal the limit of $\dfrac{\dfrac{\frac{-1}{2}(x+1)^{-\frac{3}{2}}}{\sqrt{1-\left(\frac{1}{x+1}\right)}}}{\frac{-1}{2}x^{-\frac{3}{2}}}$ ***M1M1A1A1***

Note: ***M1*** for attempting differentiation of the top and bottom, ***M1A1*** for derivative of top (only award ***M1*** if chain rule is used), ***A1*** for derivative of bottom.

$= \lim\limits_{x\to\infty} \dfrac{\left(\frac{x}{(x+1)}\right)^{\frac{3}{2}}}{\sqrt{\frac{x}{x+1}}} = \lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)$ ***M1***

Note: Accept any intermediate tidying up of correct derivative for the method mark.

$= 1$ ***A1***

[6 marks]

b. (i) $a_1 = \sqrt{2},\ a_2 = \sqrt{3}$ ***A1***

$a_n = \sqrt{n+1}$ ***A1***

(ii) $\sin\theta_n = \frac{1}{a_n} = \frac{1}{\sqrt{n+1}}$ ***A1***

Note: Allow $\theta_n = \arcsin\left(\frac{1}{a_n}\right)$ if $a_n = \sqrt{n+1}$ in b(i).

so $\theta_n = \arcsin\frac{1}{\sqrt{(n+1)}}$ ***AG***

[3 marks]

c. for $\sum\limits_{n=1}^{\infty}\arcsin\frac{1}{\sqrt{(n+1)}}$ apply the limit comparison test (since both series of positive terms) ***M1***

with $\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ ***A1***

from (a) $\lim\limits_{n\to\infty}\dfrac{\arcsin\frac{1}{\sqrt{(n+1)}}}{\frac{1}{\sqrt{n}}} = 1$, so the two series either both converge or both diverge ***M1R1***

$\sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{2}}$ diverges (as is a p-series with $p = \frac{1}{2}$) ***A1***

hence $\sum\limits_{n=1}^{\infty}\theta_n$ diverges ***A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

The sequence $\{u_n\}$ is defined by $u_n = \frac{3n+2}{2n-1}$, for $n \in \mathbb{Z}^+$.

a. Show that the sequence converges to a limit L , the value of which should be stated. [3]

b. Find the least value of the integer N such that $|u_n - L| < \varepsilon$, for all $n > N$ where [4]

(i) $\varepsilon = 0.1$;
(ii) $\varepsilon = 0.00001$.

c. For each of the sequences $\left\{\frac{u_n}{n}\right\}$, $\left\{\frac{1}{2u_n-2}\right\}$ and $\left\{(-1)^n u_n\right\}$, determine whether or not it converges. [6]

d. Prove that the series $\sum\limits_{n=1}^{\infty}(u_n - L)$ diverges. [2]

Markscheme

a. $u_n = \frac{3+\frac{2}{n}}{2-\frac{1}{n}}$ or $\frac{3}{2} + \frac{A}{2n-1}$ ***M1***

using $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ ***(M1)***

obtain $\lim\limits_{n\to\infty} u_n = \frac{3}{2} = L$ ***A1 N1***

[3 marks]

b. $u_n - L = \frac{7}{2(2n-1)}$ ***(A1)***

$|u_n - L| < \varepsilon \Rightarrow n > \frac{1}{2}\left(1 + \frac{7}{2\varepsilon}\right)$ ***(M1)***

(i) $\varepsilon = 0.1 \Rightarrow N = 18$ ***A1***

(ii) $\varepsilon = 0.00001 \Rightarrow N = 175000$ ***A1***

[4 marks]

c. $u_n \to L$ and $\frac{1}{n} \to 0$ ***M1***

$\Rightarrow \frac{u_n}{n} \to (L \times 0) = 0$, hence converges ***A1***

$2u_n - 2 \to 2L - 2 = 1 \Rightarrow \frac{1}{2u_n-2} \to 1$, hence converges ***M1A1***

Note: To award ***A1*** the value of the limit and a statement of convergence must be clearly seen for each sequence.

$(-1)^n u_n$ does not converge ***A1***

The sequence alternates (or equivalent wording) between values close to $\pm L$ ***R1***

[6 marks]

d. $u_n - L > \frac{7}{4n}$ (re: harmonic sequence) ***M1***

$\Rightarrow \sum\limits_{n=1}^{\infty}(u_n - L)$ diverges by the comparison theorem ***R1***

Note: Accept alternative methods.

[2 marks]

Examiners report

a. The “show that” in part (a) of this problem was not adequately dealt with by a significant minority of candidates and simply stating the limit and not demonstrating its existence lost marks. Part (b), whilst being possible without significant knowledge of limits, seemed to intimidate some candidates due to its unfamiliarity and the notation. Part (c) was somewhat disappointing as many candidates attempted to apply rules on the convergence of series to solve a problem that was dealing with the limits of sequences. The same confusion was seen on part (d) where also some errors in algebra prevented candidates from achieving full marks.

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The function f is defined by

$$f(x) = \begin{cases} x^2 - 2, & x < 1 \\ ax + b, & x \geqslant 1 \end{cases}$$

where a and b are real constants.

Given that both f and its derivative are continuous at $x = 1$, find the value of a and the value of b.

Markscheme

considering continuity $\lim\limits_{x \to 1^-} (x^2 - 2) = -1$ ***(M1)***

$a + b = -1$ ***(A1)***

considering differentiability $2x = a$ when $x = 1$ ***(M1)***

$\Rightarrow a = 2$ ***A1***

$b = -3$ ***A1***

[5 marks]

Examiners report

[N/A]

Let $f(x)$ be a function whose first and second derivatives both exist on the closed interval $[0,\ h]$.

Let $g(x) = f(h) - f(x) - (h-x)f'(x) - \frac{(h-x)^2}{h^2}(f(h) - f(0) - hf'(0))$.

a. State the mean value theorem for a function that is continuous on the closed interval $[a,\ b]$ and differentiable on the open interval $]a,\ b[$. [2]

b. (i) Find $g(0)$. [9]

(ii) Find $g(h)$.

(iii) Apply the mean value theorem to the function $g(x)$ on the closed interval $[0,\ h]$ to show that there exists c in the open interval $]0,\ h[$ such that $g'(c) = 0$.

(iv) Find $g'(x)$.

(v) Hence show that $-(h-c)f''(c) + \frac{2(h-c)}{h^2}(f(h) - f(0) - hf'(0)) = 0$.

(vi) Deduce that $f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(c)$.

c. Hence show that, for $h > 0$ [5]

$1 - \cos(h) \leqslant \frac{h^2}{2}$.

Markscheme

a. there exists c in the open interval $]a,\ b[$ such that ***A1***

$\frac{f(b)-f(a)}{b-a} = f'(c)$ ***A1***

Note: Open interval is required for the ***A1***.

[2 marks]

b. (i) $g(0) = f(h) - f(0) - hf'(0) - \frac{h^2}{h^2}(f(h) - f(0) - hf'(0))$

$= 0$ ***A1***

(ii) $g(h) = f(h) - f(h) - 0 - 0$

$= 0$ ***A1***

(iii) ($g(x)$ is a differentiable function since it is a combination of other differentiable functions f, f' and polynomials.)

there exists c in the open interval $]0,\ h[$ such that

$\frac{g(h)-g(0)}{h} = g'(c)$ ***A1***

$\frac{g(h)-g(0)}{h} = 0$ ***A1***

hence $g'(c) = 0$ ***AG***

(iv) $g'(x) = -f'(x) + f'(x) - (h-x)f''(x) + \frac{2(h-x)}{h^2}(f(h) - f(0) - hf'(0))$ ***A1A1***

Note: ***A1*** for the second and third terms and ***A1*** for the other terms (all terms must be seen).

$= -(h-x)f''(x) + \frac{2(h-x)}{h^2}(f(h) - f(0) - hf'(0))$

(v) putting $x = c$ and equating to zero ***M1***

$-(h-c)f''(c) + \frac{2(h-c)}{h^2}(f(h) - f(0) - hf'(0)) = g'(c) = 0$ ***AG***

(vi) $-f''(c) + \frac{2}{h^2}(f(h) - f(0) - hf'(0)) = 0$ ***A1***

since $h - c \neq 0$ ***R1***

$\frac{h^2}{2}f''(c) = f(h) - f(0) - hf'(0)$

$f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(c)$ ***AG***

[9 marks]

c. letting $f(x) = \cos(x)$ ***M1***

$f'(x) = -\sin(x) \quad f''(x) = -\cos(x)$ ***A1***

$\cos(h) = 1 + 0 - \frac{h^2}{2}\cos(c)$ ***A1***

$1 - \cos(h) = \frac{h^2}{2}\cos(c)$ ***(A1)***

since $\cos(c) \leqslant 1$ ***R1***

$1 - \cos(h) \leqslant \frac{h^2}{2}$ ***AG***

Note: Allow $f(x) = a \pm b\cos x$.

[5 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

---

a. Given that $f(x) = \ln x$, use the mean value theorem to show that, for $0 < a < b$, $\frac{b-a}{b} < \ln\frac{b}{a} < \frac{b-a}{a}$. [7]

b. Hence show that $\ln(1.2)$ lies between $\frac{1}{m}$ and $\frac{1}{n}$, where m, n are consecutive positive integers to be determined. [2]

Markscheme

a. $f'(x) = \frac{1}{x}$ ***(A1)***

using the MVT $f'(c) = \frac{f(b)-f(a)}{b-a}$ (where c lies between a and b) ***(M1)***

$f'(c) = \frac{\ln b - \ln a}{b-a}$ ***A1***

$\ln\frac{b}{a} = \ln b - \ln a$ ***(M1)***

$f'(c) = \frac{\ln\frac{b}{a}}{b-a}$

since $f'(x)$ is a decreasing function or $a < c < b \Rightarrow \frac{1}{b} < \frac{1}{c} < \frac{1}{a}$ ***R1***

$f'(b) < f'(c) < f'(a)$ ***(M1)***

$\frac{1}{b} < \frac{\ln \frac{b}{a}}{b-a} < \frac{1}{a}$ ***A1***

$\frac{b-a}{b} < \ln \frac{b}{a} < \frac{b-a}{a}$ ***AG***

[7 marks]

b. putting $b = 1.2,\ a = 1$, or equivalent ***M1***

$\frac{1}{6} < \ln 1.2 < \frac{1}{5}$ ***A1***

$(m = 6,\ n = 5)$

[2 marks]

Examiners report

a. Although many candidates achieved at least a few marks in this question, the answers revealed difficulties in setting up a proof. The Mean value theorem was poorly quoted and steps were often skipped. The conditions under which the Mean value theorem is valid were largely ignored, as were the reasoned steps towards the answer.

b. There were inequalities everywhere, without a great deal of meaning or showing progress. A number of candidates attempted to work backwards and presented the work in a way that made it difficult to follow their reasoning; in part (b) many candidates ignored the instruction 'hence' and just used GDC to find the required values; candidates that did notice the link to part a) answered this question well in general. A number of candidates guessed the answer and did not present an analytical derivation as required.

---

a. Find the radius of convergence of the series $\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)3^n}$. [6]

b. Determine whether the series $\sum\limits_{n=0}^{\infty} \left(\sqrt[3]{n^3+1} - n\right)$ is convergent or divergent. [7]

Markscheme

a. The ratio test gives

$\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim\limits_{n\to\infty}\left|\frac{(-1)^{n+1} x^{n+1}(n+1)3^n}{(n+2)3^{n+1}(-1)^n x^n}\right|$ ***M1A1***

$= \lim\limits_{n\to\infty}\left|\frac{(n+1)x}{3(n+2)}\right|$ ***A1***

$= \frac{|x|}{3}$ ***A1***

So the series converges for $\frac{|x|}{3} < 1,$ ***A1***

the radius of convergence is 3 ***A1***

Note: Do not penalise lack of modulus signs.

[6 marks]

b. $u_n = \sqrt[3]{n^3+1} - n$

$= n\left(\sqrt[3]{1+\frac{1}{n^3}}-1\right)$ ***M1A1***

$= n\left(1+\frac{1}{3n^3}-\frac{1}{9n^6}+\frac{5}{81n^9}-\ldots-1\right)$ ***A1***

using $v_n = \frac{1}{n^2}$ as the auxilliary series, ***M1***

since $\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{3}$ and $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ converges ***M1A1***

then $\sum u_n$ converges ***A1***

Note: Award ***M1A1A1M0M0A0A0*** to candidates attempting to use the integral test.

[7 marks]

Examiners report

a. Some corners were cut in applying the ratio test and some candidates tried to use the comparison test. With careful algebra finding the radius of convergence was not too difficult. Often the interval of convergence was given instead of the radius.
Part (b) was done only by the best candidates. A little algebraic manipulation together with an auxiliary series soon gave the answer.

b. Some corners were cut in applying the ratio test and some candidates tried to use the comparison test. With careful algebra finding the radius of convergence was not too difficult. Often the interval of convergence was given instead of the radius.
Part (b) was done only by the best candidates. A little algebraic manipulation together with an auxiliary series soon gave the answer.

Solve the differential equation

$\frac{dy}{dx} = \frac{y}{x}+\frac{y^2}{x^2}$ (where $x > 0$)

given that $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

Markscheme

put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ***M1A1***

the equation becomes $v + x\frac{dv}{dx} = v + v^2$ ***(A1)***

leading to $x\frac{dv}{dx} = v^2$ ***A1***

separating variables, $\int\frac{dx}{x} = \int\frac{dv}{v^2}$ ***M1A1***

hence $\ln x = -v^{-1} + C$ ***A1A1***

substituting for v, $\ln x = \frac{-x}{y} + C$ ***M1***

Note: Do not penalise absence of C at the above stages.

substituting the boundary conditions,

$0 = -\frac{1}{2} + C$ ***M1***

$C = \frac{1}{2}$ ***A1***

the solution is $\ln x = \frac{-x}{y} + \frac{1}{2}$ ***(A1)*** leading to $y = \frac{2x}{1-2\ln x}$ (or equivalent form) ***A1***

Note: Candidates are not required to note that $x \neq \sqrt{\mathrm{e}}$.

[13 marks]

Examiners report

Many candidates were able to make a reasonable attempt at this question with many perfect solutions seen.

---

a. Find the set of values of *k* for which the improper integral $\int_2^\infty \frac{\mathrm{d}x}{x(\ln x)^k}$ converges. [6]

b. Show that the series $\sum_{r=2}^{\infty} \frac{(-1)^r}{r\ln r}$ is convergent but not absolutely convergent. [5]

Markscheme

a. consider the limit as $R \to \infty$ of the (proper) integral

$\int_2^R \frac{\mathrm{d}x}{x(\ln x)^k}$ ***(M1)***

substitute $u = \ln x,\ \mathrm{d}u = \frac{1}{x}\mathrm{d}x$ ***(M1)***

obtain $\int_{\ln 2}^{\ln R} \frac{1}{u^k}\mathrm{d}u = \left[-\frac{1}{k-1}\frac{1}{u^{k-1}}\right]_{\ln 2}^{\ln R}$ ***A1***

Note: Ignore incorrect limits or omission of limits at this stage.

or $[\ln u]_{\ln 2}^{\ln R}$ if *k* = 1 ***A1***

Note: Ignore incorrect limits or omission of limits at this stage.

because $\ln R$ (and $\ln\ln R$) $\to \infty$ as $R \to \infty$ ***(M1)***

converges in the limit if *k* > 1 ***A1***

[6 marks]

b. C: terms $\to$ 0 as $r \to \infty$ ***A1***

$|u_{r+1}| < |u_r|$ for all r ***A1***

convergence by alternating series test ***R1***

AC: $(x\ln x)^{-1}$ is positive and decreasing on $[2,\ \infty)$ ***A1***

not absolutely convergent by integral test using part (a) for *k* = 1 ***R1***

[5 marks]

Examiners report

a. A good number of candidates were able to find the integral in part (a) although the vast majority did not consider separately the integral when *k* = 1. Many candidates did not explicitly set a limit for the integral to let this limit go to infinity in the anti – derivative and it seemed that some candidates were "substituting for infinity". This did not always prevent candidates finding a correct final answer but the lack of good technique is a concern. In part (b) many candidates seemed to have some knowledge of the relevant test for convergence but this test was not always rigorously applied. In showing that the series was not absolutely convergent candidates were often not clear in showing that the function being tested had to meet a number of criteria and in so doing lost marks.

b. A good number of candidates were able to find the integral in part (a) although the vast majority did not consider separately the integral when $k = 1$. Many candidates did not explicitly set a limit for the integral to let this limit go to infinity in the anti – derivative and it seemed that some candidates were "substituting for infinity". This did not always prevent candidates finding a correct final answer but the lack of good technique is a concern. In part (b) many candidates seemed to have some knowledge of the relevant test for convergence but this test was not always rigorously applied. In showing that the series was not absolutely convergent candidates were often not clear in showing that the function being tested had to meet a number of criteria and in so doing lost marks.

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$ is convergent or divergent.

Markscheme

Consider

$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{10}}{10^{n+1}} \times \frac{10^n}{n^{10}}$ ***M1A1***

$= \frac{1}{10}\left(1 + \frac{1}{n}\right)^{10}$ ***A1***

$\to \frac{1}{10}$ as $n \to \infty$ ***A1***

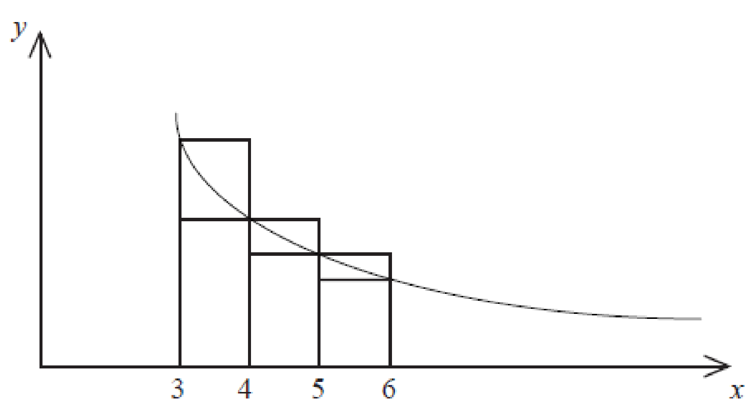
$\frac{1}{10} < 1$ ***R1***

So by the Ratio Test the series is convergent. ***R1***

[6 marks]

Examiners report

Most candidates used the Ratio Test successfully to establish convergence. Candidates who attempted to use Cauchy's (Root) Test were often less successful although this was a valid method.



The diagram shows part of the graph of $y = \frac{1}{x^3}$ together with line segments parallel to the coordinate axes.

(a) Using the diagram, show that $\frac{1}{4^3}+\frac{1}{5^3}+\frac{1}{6^3}+\ldots<\int_3^{\infty}\frac{1}{x^3}\mathrm{d}x<\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots$.

(b) **Hence** find upper and lower bounds for $\sum_{n=1}^{\infty}\frac{1}{n^3}$.

Markscheme

(a) The area under the curve is sandwiched between the sum of the areas of the lower rectangles and the upper rectangles. ***M2***

Therefore

$1\times\frac{1}{4^3}+1\times\frac{1}{5^3}+1\times\frac{1}{6^3}+\ldots<\int_3^{\infty}\frac{\mathrm{d}x}{x^3}<1\times\frac{1}{3^3}+1\times\frac{1}{4^3}+1\times\frac{1}{5^3}+\ldots$ ***A1***

which leads to the printed result.

[3 marks]

(b) We note first that

$\int_3^{\infty}\frac{\mathrm{d}x}{x^3}=\left[-\frac{1}{2x^2}\right]_3^{\infty}=\frac{1}{18}$ ***M1A1***

Consider first

$\sum_{n=1}^{\infty}\frac{1}{n^3}=1+\frac{1}{2^3}+\frac{1}{3^3}+\left(\frac{1}{4^3}+\frac{1}{5^3}+\frac{1}{6^3}+\ldots\right)$ ***M1A1***

$<1+\frac{1}{8}+\frac{1}{27}+\frac{1}{18}$ ***M1A1***

$=\frac{263}{216}$ (1.22) (which is an upper bound) ***A1***

$\sum_{n=1}^{\infty}\frac{1}{n^3}=1+\frac{1}{2^3}+\left(\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\ldots\right)$ ***M1A1***

$>1+\frac{1}{8}+\frac{1}{18}$ ***M1A1***

$=\frac{85}{72}\left(\frac{255}{216}\right)$ (1.18) (which is a lower bound) ***A1***

[12 marks]

Total [15 marks]

Examiners report

Many candidates failed to give a convincing argument to establish the inequality. In (b), few candidates progressed beyond simply evaluating the integral.

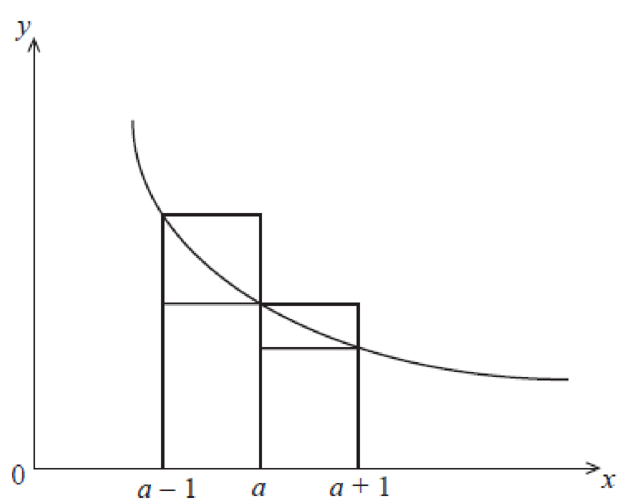


Figure 1

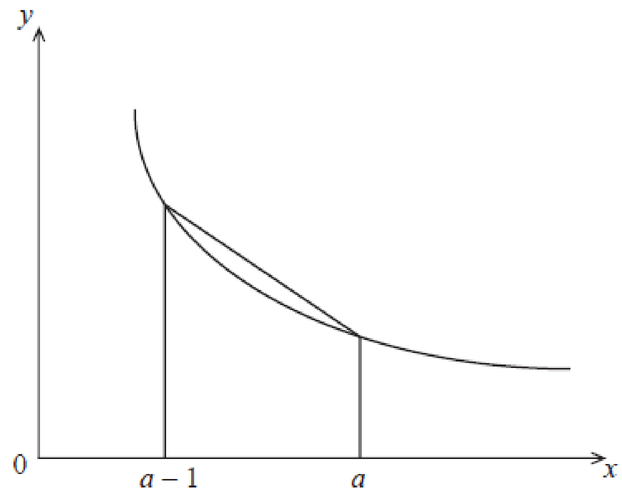


Figure 2

a. Figure 1 shows part of the graph of $y = \frac{1}{x}$ together with line segments parallel to the coordinate axes. [9]

(i) By considering the areas of appropriate rectangles, show that

$$\frac{2a+1}{a(a+1)} < \ln\left(\frac{a+1}{a-1}\right) < \frac{2a-1}{a(a-1)}.$$

(ii) Hence find lower and upper bounds for $\ln(1.2)$.

b. An improved upper bound can be found by considering Figure 2 which again shows part of the graph of $y = \frac{1}{x}$. [5]

(i) By considering the areas of appropriate regions, show that

$$\ln\left(\frac{a}{a-1}\right) < \frac{2a-1}{2a(a-1)}.$$

(ii) Hence find an upper bound for $\ln(1.2)$.

Markscheme

a. (i) the area under the curve between $a - 1$ and $a + 1$

$= \int_{a-1}^{a+1} \frac{dx}{x}$ ***M1***

$= [\ln x]_{a-1}^{a+1}$ ***A1***

$= \ln\left(\frac{a+1}{a-1}\right)$ ***A1***

lower sum $= \frac{1}{a} + \frac{1}{a+1}$ ***M1A1***

$= \frac{2a+1}{a(a+1)}$ ***AG***

upper sum $= \frac{1}{a-1} + \frac{1}{a}$ ***A1***

$= \frac{2a-1}{a(a-1)}$ ***AG***

it follows that

$\frac{2a+1}{a(a+1)} < \ln\left(\frac{a+1}{a-1}\right) < \frac{2a-1}{a(a-1)}$

because the area of the region under the curve lies between the areas of the regions defined by the lower and upper sums ***R1***

(ii) putting

$\left(\frac{a+1}{a-1}=1.2\right)\Rightarrow a=11$ ***A1***

therefore, $\text{UB}=\frac{21}{110}(=0.191),\ \text{LB}=\frac{23}{132}(=0.174)$ ***A1***

[9 marks]

b. (i) the area under the curve between *a* – 1 and *a*

$=\int_{a-1}^{a}\frac{\mathrm{d}x}{x}$ ***A1***

$=[\ln x]_{a-1}^{a}=\ln\left(\frac{a}{a-1}\right)$

attempt to find area of trapezium ***M1***

area of trapezoidal "upper sum" $=\frac{1}{2}\left(\frac{1}{a-1}+\frac{1}{a}\right)$ or equivalent ***A1***

$=\frac{2a-1}{2a(a-1)}$

it follows that $\ln\left(\frac{a}{a-1}\right)<\frac{2a-1}{2a(a-1)}$ ***AG***

(ii) putting

$\left(\frac{a}{a-1}=1.2\right)\Rightarrow a=6$ ***A1***

therefore, $\text{UB}=\frac{11}{60}(=0.183)$ ***A1***

[5 marks]

Examiners report

a. Many candidates made progress with this problem. This was pleasing since whilst being relatively straightforward it was not a standard problem. There were still some candidates who did not use the definite integral correctly to find the area under the curve in part (a) and part (b). Also candidates should take care to show all the required working in a "show that" question, even when demonstrating familiar results. The ability to find upper and lower bounds was often well done in parts (a) (ii) and (b) (ii).

b. Many candidates made progress with this problem. This was pleasing since whilst being relatively straightforward it was not a standard problem. There were still some candidates who did not use the definite integral correctly to find the area under the curve in part (a) and part (b). Also candidates should take care to show all the required working in a "show that" question, even when demonstrating familiar results. The ability to find upper and lower bounds was often well done in parts (a) (ii) and (b) (ii).

The function *f* is defined by $f(x)=\begin{cases}e^{-x^3}(-x^3+2x^2+x), x\le 1\\ ax+b, x>1\end{cases}$, where a and b are constants.

a. Find the exact values of a and b if f is continuous and differentiable at $x=1$. [8]

b. (i) Use Rolle's theorem, applied to f, to prove that $2x^4 - 4x^3 - 5x^2 + 4x + 1 = 0$ has a root in the interval $]-1, 1[$. [7]

(ii) Hence prove that $2x^4 - 4x^3 - 5x^2 + 4x + 1 = 0$ has at least two roots in the interval $]-1, 1[$.

Markscheme

a. $\lim\limits_{x \to 1^-} \mathrm{e}^{-x^2}\left(-x^3 + 2x^2 + x\right) = \lim\limits_{x \to 1^+} (ax + b) \quad (= a + b)$ ***M1***

$2\mathrm{e}^{-1} = a + b$ ***A1***

differentiability: attempt to differentiate **both** expressions ***M1***

$f'(x) = -2x\mathrm{e}^{-x^2}\left(-x^3 + 2x^2 + x\right) + \mathrm{e}^{-x^2}\left(-3x^2 + 4x + 1\right) \quad (x < 1)$ ***A1***

(or $f'(x) = \mathrm{e}^{-x^2}\left(2x^4 - 4x^3 - 5x^2 + 4x + 1\right)$)

$f'(x) = a \quad (x > 1)$ ***A1***

substitute $x = 1$ in **both** expressions and equate

$-2\mathrm{e}^{-1} = a$ ***A1***

substitute value of a and find $b = 4\mathrm{e}^{-1}$ ***M1A1***

[8 marks]

b. (i) $f'(x) = \mathrm{e}^{-x^2}\left(2x^4 - 4x^3 - 5x^2 + 4x + 1\right)$ (for $x \leqslant 1$) ***M1***

$f(1) = f(-1)$ ***M1***

Rolle's theorem statement ***(A1)***

by Rolle's Theorem, $f'(x)$ has a zero in $]-1, 1[$ ***R1***

hence quartic equation has a root in $]-1, 1[$ ***AG***

(ii) let $g(x) = 2x^4 - 4x^3 - 5x^2 + 4x + 1$.

$g(-1) = g(1) < 0$ and $g(0) > 0$ ***M1***

as g is a polynomial function it is continuous in $[-1, 0]$ and $[0,\ 1]$. ***R1***

(or g is a polynomial function continuous in any interval of real numbers)

then the graph of g must cross the x-axis at least once in $]-1, 0[$ ***R1***

and at least once in $]0, 1[$.

[7 marks]

Examiners report

a. [N/A]

b. [N/A]

(a) Using the Maclaurin series for $(1 + x)^n$, write down and simplify the Maclaurin series approximation for $(1 - x^2)^{-\frac{1}{2}}$ as far as the term in x^4

(b) Use your result to show that a series approximation for arccos x is

$$\arccos x \approx \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5.$$

(c) Evaluate $\lim\limits_{x \to 0} \frac{\frac{\pi}{2} - \arccos(x^2) - x^2}{x^6}$.

(d) Use the series approximation for $\arccos x$ to find an approximate value for

$$\int_0^{0.2} \arccos\left(\sqrt{x}\right) \mathrm{d}x,$$

giving your answer to 5 decimal places. Does your answer give the actual value of the integral to 5 decimal places?

Markscheme

(a) using or obtaining $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$ ***(M1)***

$(1-n^2)^{-\frac{1}{2}} = 1 + (-x^2) \times \left(-\frac{1}{2}\right) + \frac{(-x^2)^2}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) + \ldots$ ***(A1)***

$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \ldots$ ***A1***

[3 marks]

(b) integrating, and changing sign

$\arccos x = -x - \frac{1}{6}x^3 - \frac{3}{40}x^5 + C + \ldots$ ***M1A1***

put $x = 0$,

$\frac{\pi}{2} = C$ ***M1***

$\left(\arccos x \approx \frac{\pi}{2} - x - \frac{1}{6}x^3 - \frac{3}{40}x^5\right)$ ***AG***

[3 marks]

(c) **EITHER**

using $\arccos x^2 \approx \frac{\pi}{2} - x^2 - \frac{1}{6}x^6 - \frac{3}{40}x^{10}$ ***M1A1***

$\lim\limits_{x\to 0} \frac{\frac{\pi}{2} - \arccos x^2 - x^2}{x^6} = \lim\limits_{x\to 0} \frac{\frac{x^6}{6} + \text{higher powers}}{x^6}$ ***M1A1***

$= \frac{1}{6}$ ***A1***

OR

using l'Hôpital's Rule ***M1***

$\text{limit} = \lim\limits_{x\to 0} \frac{\frac{1}{\sqrt{1-x^4}} \times 2x - 2x}{6x^5}$ ***M1***

$= \lim\limits_{x\to 0} \frac{\frac{1}{\sqrt{1-x^4}} - 1}{3x^4}$ ***A1***

$= \lim\limits_{x\to 0} \frac{-\frac{1}{2} \times \frac{1}{(1-x^4)^{3/2}} \times -4x^3}{12x^3}$ ***M1***

$= \frac{1}{6}$ ***A1***

[5 marks]

(d) $\int_0^{0.2} \arccos \sqrt{x} \mathrm{d}x \approx \int_0^{0.2} \left(\frac{\pi}{2} - x^{\frac{1}{2}} - \frac{1}{6}x^{\frac{3}{2}} - \frac{3}{40}x^{\frac{5}{2}}\right) \mathrm{d}x$ ***M1***

$= \left[\frac{\pi}{2}x - \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{15}x^{\frac{5}{2}} - \frac{3}{140}x^{\frac{7}{2}}\right]_0^{0.2}$ ***(A1)***

$= \frac{\pi}{2} \times 0.2 - \frac{2}{3} \times 0.2^{\frac{3}{2}} - \frac{1}{15} \times 0.2^{\frac{5}{2}} - \frac{3}{140} \times 0.2^{\frac{7}{2}}$ ***(A1)***

= 0.25326 (to 5 decimal places) ***A1***

Note: Accept integration of the series approximation using a GDC.

using a GDC, the actual value is 0.25325 ***A1***

so the approximation is not correct to 5 decimal places ***R1***

[6 marks]

Total [17 marks]

Examiners report

Many candidates ignored the instruction in the question to use the series for $(1+x)^n$ to deduce the series for $(1-x^2)^{-1/2}$ and attempted instead to obtain it by successive differentiation. It was decided at the standardisation meeting to award full credit for this method although in the event the algebra proved to be too difficult for many. Many candidates used l'Hopital's Rule in (c) – this was much more difficult algebraically than using the series and it usually ended unsuccessfully. Candidates should realise that if a question on evaluating an indeterminate limit follows the determination of a Maclaurin series then it is likely that the series will be helpful in evaluating the limit. Part (d) caused problems for many candidates with algebraic errors being common. Many candidates failed to realise that the best way to find the exact value of the integral was to use the calculator.

(a) Using l'Hopital's Rule, show that $\lim\limits_{x\to\infty} x\mathrm{e}^{-x} = 0$.

(b) Determine $\int_0^a x\mathrm{e}^{-x}\mathrm{d}x$.

(c) Show that the integral $\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x$ is convergent and find its value.

Markscheme

(a) $\lim\limits_{x\to\infty} \frac{x}{\mathrm{e}^x} = \lim\limits_{x\to\infty} \frac{1}{\mathrm{e}^x}$ ***M1A1***

$= 0$ ***AG***

[2 marks]

(b) Using integration by parts ***M1***

$\int_0^a x\mathrm{e}^{-x}\mathrm{d}x = [-x\mathrm{e}^{-x}]_0^a + \int_0^a \mathrm{e}^{-x}\mathrm{d}x$ ***A1A1***

$= -a\mathrm{e}^{-a} - [\mathrm{e}^{-x}]_0^a$ ***A1***

$= 1 - a\mathrm{e}^{-a} - \mathrm{e}^{-a}$ ***A1***

[5 marks]

(c) Since e^{-a} and $a\mathrm{e}^{-a}$ are both convergent (to zero), the integral is convergent. ***R1***

Its value is 1. ***A1***

[2 marks]

Total [9 marks]

Examiners report

Most candidates made a reasonable attempt at (a). In (b), however, it was disappointing to note that some candidates were unable to use integration by parts to perform the integration. In (c), while many candidates obtained the correct value of the integral, proof of its convergence was often unconvincing.

---

a. By successive differentiation find the first four non-zero terms in the Maclaurin series for $f(x) = (x + 1)\ln(1 + x) - x$. [11]

b. Deduce that, for $n \geqslant 2$, the coefficient of x^n in this series is $(-1)^n \frac{1}{n(n-1)}$. [1]

c. By applying the ratio test, find the radius of convergence for this Maclaurin series. [6]

Markscheme

a. $f(x) = (x + 1)\ln(1 + x) - x \quad f(0) = 0$ ***A1***

$f'(x) = \ln(1 + x) + \frac{x+1}{1+x} - 1 \ (= \ln(1 + x)) \quad f'(0) = 0$ ***M1A1A1***

$f''(x) = (1 + x)^{-1} \quad f''(0) = 1$ ***A1A1***

$f'''(x) = -(1 + x)^{-2} \quad f'''(0) = -1$ ***A1***

$f^{(4)}(x) = 2(1 + x)^{-3} \quad f^{(4)}(0) = 2$ ***A1***

$f^{(5)}(x) = -3 \times 2(1 + x)^{-4} \quad f^{(5)}(0) = -3 \times 2$ ***A1***

$f(x) = \frac{x^2}{2!} - \frac{1x^3}{3!} + \frac{2x^4}{4!} - \frac{6x^5}{5!} \ldots$ ***M1A1***

$f(x) = \frac{x^2}{1\times2} - \frac{x^3}{2\times3} + \frac{x^4}{3\times4} - \frac{x^5}{4\times5} \ldots$

$f(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} \ldots$

Note: Allow follow through from the first error in a derivative (provided future derivatives also include the chain rule), no follow through after a second error in a derivative.

[11 marks]

b. $f^{(n)}(0) = (-1)^n (n - 2)!$ So coefficient of $x^n = (-1)^n \frac{(n-2)!}{n!}$ ***A1***

coefficient of x^n is $(-1)^n \frac{1}{n(n-1)}$ ***AG***

[1 mark]

c. applying the ratio test to the series of absolute terms

$\lim\limits_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)n}}{\frac{|x|^n}{n(n-1)}}$ ***M1A1***

$= \lim\limits_{n\to\infty} |x| \frac{(n-1)}{(n+1)}$ ***A1***

$= |x|$ ***A1***

so for convergence $|x| < 1$, giving radius of convergence as 1 ***(M1)A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]

Find $\lim\limits_{x \to \frac{1}{2}} \left(\frac{\left(\frac{1}{4} - x^2\right)}{\cot \pi x} \right)$.

Markscheme

using l'Hôpital's Rule ***(M1)***

$\lim\limits_{x \to \frac{1}{2}} \left(\frac{\left(\frac{1}{4} - x^2\right)}{\cot \pi x} \right) = \lim\limits_{x \to \frac{1}{2}} \left[\frac{-2}{-\pi \text{cosec}^2 \pi x} \right]$ ***A1A1***

$= \frac{-1}{-\pi \text{cosec}^2 \frac{\pi}{2}} = \frac{1}{\pi}$ ***(M1)A1***

[5 marks]

Examiners report

This question was accessible to the vast majority of candidates, who recognised that L'Hôpital's rule was required. However, some candidates omitted the factor π in the differentiation of $\cot \pi x$. Some candidates replaced $\cot \pi x$ by $\cos \pi x / \sin \pi x$, which is a valid method but the extra algebra involved often led to an incorrect answer. Many fully correct solutions were seen.

Let $f(x) = 2x + |x|$, $x \in \mathbb{R}$.

a. Prove that *f* is continuous but not differentiable at the point (0, 0) . [7]

b. Determine the value of $\int_{-a}^{a} f(x)\mathrm{d}x$ where $a > 0$. [3]

Markscheme

a. we note that $f(0) = 0,\ f(x) = 3x$ for $x > 0$ and $f(x) = x$ for $x < 0$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x = 0$ ***M1A1***

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} 3x = 0$ ***A1***

since $f(0) = 0$, the function is continuous when *x* = 0 ***AG***

$\lim\limits_{x \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{x \to 0^-} \frac{h}{h} = 1$ ***M1A1***

$\lim\limits_{x \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim\limits_{x \to 0^+} \frac{3h}{h} = 3$ ***A1***

these limits are unequal ***R1***

so *f* is not differentiable when *x* = 0 ***AG***

[7 marks]

b. $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{-a}^{0} x\mathrm{d}x + \int_{0}^{a} 3x\mathrm{d}x$ ***M1***

$= \left[\frac{x^2}{2}\right]_{-a}^{0} + \left[\frac{3x^2}{2}\right]_{0}^{a}$ ***A1***

$= a^2$ ***A1***

[3 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the curve $y = \frac{1}{x},\ x > 0$.

Let $U_n = \sum_{r=1}^{n} \frac{1}{r} - \ln n$.

a. By drawing a diagram and considering the area of a suitable region under the curve, show that for $r > 0$, [4]

$$\frac{1}{r+1} < \ln\left(\frac{r+1}{r}\right) < \frac{1}{r}.$$

b.i. Hence, given that n is a positive integer greater than one, show that [3]

$\sum_{r=1}^{n} \frac{1}{r} > \ln(1+n);$

b.ii. Hence, given that n is a positive integer greater than one, show that [3]

$\sum_{r=1}^{n} \frac{1}{r} < 1 + \ln n.$

c.i. Hence, given that n is a positive integer greater than one, show that [1]

$U_n > 0;$

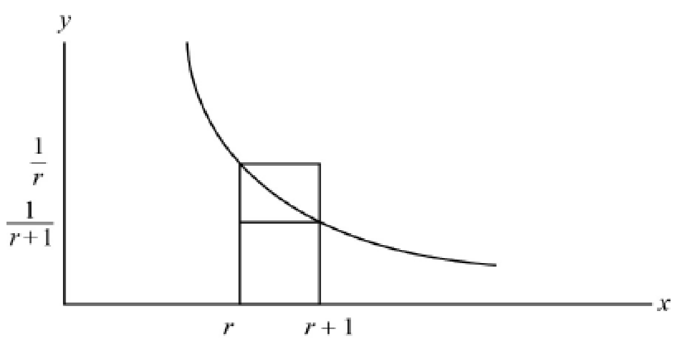
c.ii. Hence, given that n is a positive integer greater than one, show that [3]

$U_{n+1} < U_n.$

d. Explain why these two results prove that $\{U_n\}$ is a convergent sequence. [1]

Markscheme

a.



A1

Note: Curve, both rectangles and correct $x$values required.

area of rectangles $\frac{1}{r}$ and $\frac{1}{1+r}$ ***A1***

Note: Correct values on the y-axis are sufficient evidence for this mark if not otherwise indicated.

in the above diagram, the area below the curve between $x = r$ and $x = r + 1$ is between the areas of the larger and smaller rectangle

or $\frac{1}{r+1} < \int\limits_r^{r+1} \frac{\mathrm{d}x}{x} < \frac{1}{r}$ ***(R1)***

integrating, $\int_r^{r+1} \frac{\mathrm{d}x}{x} = [\ln x]_r^{r+1} \quad (= \ln(r+1) - \ln(r))$ ***A1***

$\frac{1}{r+1} < \ln\left(\frac{r+1}{r}\right) < \frac{1}{r}$ ***AG***

[4 marks]

b.i. summing the right-hand part of the above inequality from $r = 1$ to $r = n$,

$\sum\limits_{r=1}^{n} \frac{1}{r} > \sum\limits_{r=1}^{n} \ln\left(\frac{r+1}{r}\right)$ ***M1***

$= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ldots + \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n+1}{n}\right)$ ***(A1)***

EITHER

$= \ln\left(\frac{2}{1} \times \frac{3}{2} \times \ldots \times \frac{n}{n-1} \times \frac{n+1}{n}\right)$ ***A1***

OR

$\ln 2 - \ln 1 + \ln 3 - \ln 2 + \ldots + \ln(n+1) - \ln(n)$ ***A1***

$= \ln(n+1)$ ***AG***

[3 marks]

b.ii. $\sum\limits_{r=1}^{n} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < 1 + \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ldots + \ln\left(\frac{n}{n-1}\right)$ ***M1A1A1***

$\left(1 + \sum\limits_{r=1}^{n-1} \frac{1}{r+1} < 1 + \sum\limits_{r=1}^{n-1} \ln\left(\frac{r+1}{r}\right)\right)$

Note: ***M1*** is for using the correct inequality from (a), ***A1*** for both sides beginning with 1, ***A1*** for completely correct expression.

Note: The 1 might be added after the sums have been calculated.

$= 1 + \ln n$ ***AG***

[3 marks]

c.i. from (b)(i) $U_n > \ln(1+n) - \ln n > 0$ ***A1***

[1 mark]

c.ii. $U_{n+1} - U_n = \sum\limits_{r=1}^{n+1} \frac{1}{r} - \ln(n+1) - \sum\limits_{r=1}^{n} \frac{1}{r} + \ln n$ ***M1***

$= \frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right)$ ***A1***

< 0 (using the result proved in (a)) ***A1***

$U_{n+1} < U_n$ ***AG***

[3 marks]

d. it follows from the two results that $\{U_n\}$ cannot be divergent either in the sense of tending to $-\infty$ or oscillating therefore it must be convergent

R1

Note: Accept the use of the result that a bounded (monotonically) decreasing sequence is convergent (allow "positive, decreasing sequence").

[1 mark]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]
c.i. [N/A]
c.ii. [N/A]
d. [N/A]

---

a. Find the value of $\lim\limits_{x\to 1}\left(\frac{\ln x}{\sin 2\pi x}\right)$. [3]

b. By using the series expansions for e^{x^2} and $\cos x$ evaluate $\lim\limits_{x\to 0}\left(\frac{1-e^{x^2}}{1-\cos x}\right)$.. [7]

Markscheme

a. Using l'Hopital's rule,

$\lim\limits_{x\to 1}\left(\frac{\ln x}{\sin 2\pi x}\right) = \lim\limits_{x\to 1}\left(\frac{\frac{1}{x}}{2\pi\cos 2\pi x}\right)$ ***M1A1***

$= \frac{1}{2\pi}$ ***A1***

[3 marks]

b. $\lim\limits_{x\to 0}\left(\frac{1-e^{x^2}}{1-\cos x}\right) = \lim\limits_{x\to 0}\left(\frac{1-\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\ldots\right)}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)}\right)$ ***M1A1A1***

Note: Award ***M1*** for evidence of using the two series.

$= \lim\limits_{x\to 0}\left(\frac{\left(-x^2-\frac{x^4}{2!}-\frac{x^6}{3!}-\ldots\right)}{\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\ldots\right)}\right)$ ***A1***

EITHER

$= \lim\limits_{x\to 0}\left(\frac{\left(-1-\frac{x^2}{2!}-\frac{x^4}{3!}-\ldots\right)}{\left(\frac{1}{2!}-\frac{x^2}{4!}+\ldots\right)}\right)$ ***M1A1***

$= \frac{-1}{\frac{1}{2}} = -2$ ***A1***

OR

$= \lim\limits_{x\to 0}\left(\frac{\left(-2x-\frac{4x^3}{2!}-\frac{6x^5}{3!}-\ldots\right)}{\left(\frac{2x}{2!}-\frac{4x^3}{4!}+\ldots\right)}\right)$ ***M1A1***

$$= \lim_{x\to 0}\left(\frac{\left(-2-\frac{4x^2}{2!}-\frac{6x^4}{3!}-\ldots\right)}{\left(1-\frac{4x^2}{4!}+\ldots\right)}\right)$$

$= \frac{-2}{1} = -2$ ***A1***

[7 marks]

Examiners report

a. Part (a) was well done but too often the instruction to use series in part (b) was ignored. When this hint was observed correct solutions followed.

b. Part (a) was well done but too often the instruction to use series in part (b) was ignored. When this hint was observed correct solutions followed.

A curve that passes through the point (1, 2) is defined by the differential equation

$$\frac{dy}{dx} = 2x(1 + x^2 - y) .$$

(a) (i) Use Euler's method to get an approximate value of y when $x = 1.3$, taking steps of 0.1. Show intermediate steps to four decimal places in a table.

(ii) How can a more accurate answer be obtained using Euler's method?

(b) Solve the differential equation giving your answer in the form $y = f(x)$.

Markscheme

(a)

(i) $\frac{dy}{dx} = 2x(1 + x^2 - y)$

x_i	y_i	y_i'	Δy
1	2	0	0
1.1	2	0.4620	0.0462
1.2	2.0462	0.9451	0.0945
1.3	2.1407		

M1 ***A2***

Note: Award ***A2*** for complete table.

Award ***A1*** for a reasonable attempt.

$f(1.3) = 2.14$ (accept 2.141) ***A1***

(ii) Decrease the step size ***A1***

[5 marks]

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x(1 + x^2 - y)$

$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2x(1 + x^2)$ ***M1***

Integrating factor is $\mathrm{e}^{\int 2x\mathrm{d}x} = \mathrm{e}^{x^2}$ ***M1A1***

So, $\mathrm{e}^{x^2}y = \int (2x\mathrm{e}^{x^2} + 2x\mathrm{e}^{x^2}x^2)\mathrm{d}x$ ***A1***

$= \mathrm{e}^{x^2} + x^2\mathrm{e}^{x^2} - \int 2x\mathrm{e}^{x^2}\mathrm{d}x$ ***M1A1***

$= \mathrm{e}^{x^2} + x^2\mathrm{e}^{x^2} - \mathrm{e}^{x^2} + k$

$= x^2\mathrm{e}^{x^2} + k$ ***A1***

$y = x^2 + k\mathrm{e}^{-x^2}$

$x = 1,\ y = 2 \rightarrow 2 = 1 + k\mathrm{e}^{-1}$ ***M1***

$k = \mathrm{e}$

$y = x^2 + \mathrm{e}^{1-x^2}$ ***A1***

[9 marks]

Total [14 marks]

Examiners report

Some incomplete tables spoiled what were often otherwise good solutions. Although the intermediate steps were asked to four decimal places the answer was not and the usual degree of IB accuracy was expected.

Some candidates surprisingly could not solve what was a fairly easy differential equation in part (b).

The variables *x* and *y* are related by $\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = \cos x$.

(a) Find the Maclaurin series for *y* up to and including the term in x^2 given that $y = -\frac{\pi}{2}$ when $x = 0$.

(b) Solve the differential equation given that $y = 0$ when $x = \pi$. Give the solution in the form $y = f(x)$.

Markscheme

(a) from $\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x + \cos x$, $f'(0) = 1$ ***A1***

now $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = y\sec^2 x + \frac{\mathrm{d}y}{\mathrm{d}x}\tan x - \sin x$ ***M1A1A1A1***

Note: Award ***A1*** for each term on RHS.

$\Rightarrow f''(0) = -\frac{\pi}{2}$ ***A1***

$\Rightarrow y = -\frac{\pi}{2} + x - \frac{\pi x^2}{4}$ ***A1***

[7 marks]

(b) recognition of integrating factor ***(M1)***

integrating factor is $\mathrm{e}^{\int -\tan x \mathrm{d}x}$

$= \mathrm{e}^{\ln \cos x}$ ***(A1)***

$= \cos x$ ***(A1)***

$\Rightarrow y\cos x = \int \cos^2 x \mathrm{d}x$ ***M1***

$\Rightarrow y\cos x = \frac{1}{2}\int (1+\cos 2x)\mathrm{d}x$ ***A1***

$\Rightarrow y\cos x = \frac{x}{2} + \frac{\sin 2x}{4} + k$ ***A1***

when $x = \pi,\ y = 0 \Rightarrow k = -\frac{\pi}{2}$ ***M1A1***

$\Rightarrow y\cos x = \frac{x}{2} + \frac{\sin 2x}{4} - \frac{\pi}{2}$ ***(A1)***

$\Rightarrow y = \sec x\left(\frac{x}{2} + \frac{\sin 2x}{4} - \frac{\pi}{2}\right)$ ***A1***

[10 marks]

Total [17 marks]

Examiners report

Part (a) of the question was set up in an unusual way, which caused a problem for a number of candidates as they tried to do part (b) first and then find the Maclaurin series by a standard method. Few were successful as they were usually weaker candidates and made errors in finding the solution $y = f(x)$. The majority of candidates knew how to start part (b) and recognised the need to use an integrating factor, but a number failed because they missed out the negative sign on the integrating factor, did not realise that $\mathrm{e}^{\ln \cos x} = \cos x$ or were unable to integrate $\cos^2 x$. Having said this, a number of candidates succeeded in gaining full marks on this question.

Consider the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^p + 1$ where $x \in \mathbb{R},\ x \neq 0$ and p is a positive integer, $p > 1$.

a. Solve the differential equation given that $y = -1$ when $x = 1$. Give your answer in the form $y = f(x)$. [8]

b.i. Show that the x-coordinate(s) of the points on the curve $y = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ satisfy the equation $x^{p-1} = \frac{1}{p}$. [2]

b.ii. Deduce the set of values for p such that there are two points on the curve $y = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Give a reason for your answer. [2]

Markscheme

a. **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} = x^{p-1} + \frac{1}{x}$ ***(M1)***

integrating factor $= \mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x}$ ***M1***

$= \mathrm{e}^{-\ln x}$ ***(A1)***

$= \frac{1}{x}$ ***A1***

$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = x^{p-2} + \frac{1}{x^2}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = x^{p-2} + \frac{1}{x^2}$

$\frac{y}{x} = \frac{1}{p-1}x^{p-1} - \frac{1}{x} + C$ ***A1***

Note: Condone the absence of *C*.

$y = \frac{1}{p-1}x^p + Cx - 1$

substituting $x = 1, y = -1 \Rightarrow C = -\frac{1}{p-1}$ ***M1***

Note: Award ***M1*** for attempting to find their value of *C*.

$y = \frac{1}{p-1}(x^p - x) - 1$ ***A1***

[8 marks]

METHOD 2

put $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1(A1)***

substituting, ***M1***

$x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) - vx = x^p + 1$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = x^{p-1} + \frac{1}{x}$ ***M1***

$\frac{\mathrm{d}v}{\mathrm{d}x} = x^{p-2} + \frac{1}{x^2}$

$v = \frac{1}{p-1}x^{p-1} - \frac{1}{x} + C$ ***A1***

Note: Condone the absence of *C*.

$y = \frac{1}{p-1}x^p + Cx - 1$

substituting $x = 1, y = -1 \Rightarrow C = -\frac{1}{p-1}$ ***M1***

Note: Award ***M1*** for attempting to find their value of *C*.

$y = \frac{1}{p-1}(x^p - x) - 1$ ***A1***

[8 marks]

b.i. **METHOD 1**

find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ for x

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p-1}\left(px^{p-1} - 1\right)$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow px^{p-1} - 1 = 0$ ***A1***

$px^{p-1} = 1$

Note: Award a maximum of ***M1A0*** if a candidate's answer to part (a) is incorrect.

$x^{p-1} = \frac{1}{p}$ ***AG***

METHOD 2

substitute $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and their y into the differential equation and solve for x

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow -\left(\frac{x^p - x}{p-1}\right) + 1 = x^p + 1$ ***M1***

$x^p - x = x^p - px^p$ ***A1***

$px^{p-1} = 1$

Note: Award a maximum of ***M1A0*** if a candidate's answer to part (a) is incorrect.

$x^{p-1} = \frac{1}{p}$ ***AG***

[2 marks]

b.ii. there are two solutions for x when p is odd (and $p > 1$ ***A1***

if $p - 1$ is even there are two solutions (to $x^{p-1} = \frac{1}{p}$)

and if $p - 1$ is odd there is only one solution (to $x^{p-1} = \frac{1}{p}$) ***R1***

Note: Only award the ***R1*** if both cases are considered.

[4 marks]

Examiners report

a. [N/A]
b.i. [N/A]
b.ii. [N/A]

---

a. Use the limit comparison test to prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. [5]

c. Using the Maclaurin series for $\ln(1+x)$, show that the Maclaurin series for $(1+x)\ln(1+x)$ is $x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)}$. [3]

Markscheme

a. apply the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ***M1***

$\lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n(n+1)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$ ***M1A1***

(since the limit is finite and $\neq 0$) both series do the same ***R1***

we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and hence $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ also converges ***R1AG***

[5 marks]

c. $(1+x)\ln(1+x) = (1+x)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots\right)$ ***A1***

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots\right) + \left(x^2 - \frac{x3}{2} + \frac{x^4}{3} - \frac{x^5}{4} \ldots\right)$

EITHER

$= x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n}$ ***A1***

$= x + \sum_{n=1}^{\infty} (-1)^{n+1} x^{n+1} \left(\frac{-1}{n+1} + \frac{1}{n}\right)$ ***M1***

OR

$x + \left(1 - \frac{1}{2}\right)x^2 - \left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(\frac{1}{3} - \frac{1}{4}\right)x^4 - \ldots$ ***A1***

$= x + \sum_{n=1}^{\infty} (-1)^{n+1} x^{n+1} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ ***M1***

$= x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{n(n+1)}$ ***AG***

[3 marks]

Examiners report

a. Candidates and teachers need to be aware that the Limit comparison test is distinct from the comparison test. Quite a number of candidates lost most of the marks for this part by doing the wrong test.

Some candidates failed to state that because the result was finite and not equal to zero then the two series converge or diverge together. Others forgot to state, with a reason, that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

c. Candidates and teachers need to be aware that the Limit comparison test is distinct from the comparison test. Quite a number of candidates lost most of the marks for this part by doing the wrong test.

Some candidates failed to state that because the result was finite and not equal to zero then the two series converge or diverge together. Others forgot to state, with a reason, that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In part (b) finding the partial fractions was well done. The second part involving the use of telescoping series was less well done, and students were clearly not as familiar with this technique as with some others.

Part (c) was the least well done of all the questions. It was expected that students would use explicitly the result from the first part of 4(b) or show it once again in order to give a complete answer to this question, rather than just assuming that a pattern spotted in the first few terms would continue.

Candidates need to be informed that unless specifically told otherwise they may use without proof any of the Maclaurin expansions given in the Information Booklet. There were many candidates who lost time and gained no marks by trying to derive the expansion for $\ln(1+x)$.

In this question you may assume that $\arctan x$ is continuous and differentiable for $x \in \mathbb{R}$.

a. Consider the infinite geometric series [1]

$$1 - x^2 + x^4 - x^6 + \ldots \quad |x| < 1.$$

Show that the sum of the series is $\frac{1}{1+x^2}$.

b. Hence show that an expansion of $\arctan x$ is $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ [4]

c. f is a continuous function defined on $[a,\ b]$ and differentiable on $]a,\ b[$ with $f'(x) > 0$ on $]a,\ b[$. [4]

Use the mean value theorem to prove that for any $x,\ y \in [a,\ b]$, if $y > x$ then $f(y) > f(x)$.

d. (i) Given $g(x) = x - \arctan x$, prove that $g'(x) > 0$, for $x > 0$. [4]

(ii) Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$.

e. Use the result from part (c) to prove that $\arctan x > x - \frac{x^3}{3}$, for $x > 0$. [5]

f. Hence show that $\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$. [4]

Markscheme

a. $r = -x^2, \quad S = \frac{1}{1+x^2}$ ***A1AG***

[1 mark]

b. $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

EITHER

$\int \frac{1}{1+x^2}\mathrm{d}x = \int 1 - x^2 + x^4 - x^6 + \ldots \mathrm{d}x$ ***M1***

$\arctan x = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***A1***

Note: Do not penalize the absence of c at this stage.

when $x = 0$ we have $\arctan 0 = c$ hence $c = 0$ ***M1A1***

OR

$\int_0^x \frac{1}{1+t^2}\mathrm{d}t = \int_0^x 1 - t^2 + t^4 - t^6 + \ldots \mathrm{d}t$ ***M1A1A1***

Note: Allow x as the variable as well as the limit.

M1 for knowing to integrate, ***A1*** for each of the limits.

$[\arctan t]_0^x = \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right]_0^x$ ***A1***

hence $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ***AG***

[4 marks]

c. applying the MVT to the function f on the interval $[x,\ y]$ ***M1***

$\frac{f(y)-f(x)}{y-x} = f'(c) \quad (\text{for some } c \in]x,\ y[)$ ***A1***

$\frac{f(y)-f(x)}{y-x} > 0 \quad (\text{as } f'(c) > 0)$ ***R1***

$f(y) - f(x) > 0$ as $y > x$ ***R1***

$\Rightarrow f(y) > f(x)$ ***AG***

Note: If they use x rather than c they should be awarded ***M1A0R0***, but could get the next ***R1***.

[4 marks]

d. (i) $g(x) = x - \arctan x \Rightarrow g'(x) = 1 - \frac{1}{1+x^2}$ ***A1***

this is greater than zero because $\frac{1}{1+x^2} < 1$ ***R1***

so $g'(x) > 0$ ***AG***

(ii) (g is a continuous function defined on $[0,\ b]$ and differentiable on $]0,\ b[$ with $g'(x) > 0$ on $]0,\ b[$ for all $b \in \mathbb{R}$)

(If $x \in [0,\ b]$ then) from part (c) $g(x) > g(0)$ ***M1***

$x - \arctan x > 0 \Rightarrow \arctan x < x$ ***M1***

(as b can take any positive value it is true for all $x > 0$) ***AG***

[4 marks]

e. let $h(x) = \arctan x - \left(x - \frac{x^3}{3}\right)$ ***M1***

(h is a continuous function defined on $[0,\ b]$ and differentiable on $]0,\ b[$ with $h'(x) > 0$ on $]0,\ b[$)

$h'(x) = \frac{1}{1+x^2} - (1 - x^2)$ ***A1***

$= \frac{1-(1-x^2)(1+x^2)}{1+x^2} = \frac{x^4}{1+x^2}$ ***M1A1***

$h'(x) > 0$ hence (for $x \in [0,\ b]$) $h(x) > h(0)(= 0)$ ***R1***

$\Rightarrow \arctan x > x - \frac{x^3}{3}$ ***AG***

Note: Allow correct working with $h(x) = x - \frac{x^3}{3} - \arctan x$.

[5 marks]

f. use of $x - \frac{x^3}{3} < \arctan x < x$ ***M1***

choice of $x = \frac{1}{\sqrt{3}}$ ***A1***

$\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ ***M1***

$\frac{8}{9\sqrt{3}} < \frac{\pi}{6} < \frac{1}{\sqrt{3}}$ ***A1***

Note: Award final ***A1*** for a correct inequality with a single fraction on each side that leads to the final answer.

$\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$ ***AG***

[4 marks]

Total [22 marks]

Examiners report

a. Most candidates picked up this mark for realizing the common ratio was $-x^2$.

b. Quite a few candidates did not recognize the importance of 'hence' in this question, losing a lot of time by trying to work out the terms from first principles.

Of those who integrated the formula from part (a) only a handful remembered to include the '$+c$' term, and to verify that this must be equal to zero.

c. Most candidates were able to achieve some marks on this question. The most commonly lost mark was through not stating that the inequality was unchanged when multiplying by $y - x$ as $y > x$.

d. The first part of this question proved to be very straightforward for the majority of candidates.

In (ii) very few realized that they had to replace the lower variable in the formula from part (c) by zero.

e. Candidates found this part difficult, failing to spot which function was required.

f. Many candidates, even those who did not successfully complete (d) (ii) or (e), realized that these parts gave them the necessary inequality.

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{1+x^2}\right) y = x^2$, given that $y = 2$ when $x = 0$.

a. Show that $1 + x^2$ is an integrating factor for this differential equation. [5]

b. Hence solve this differential equation. Give the answer in the form $y = f(x)$. [6]

Markscheme

a. **METHOD 1**

attempting to find an integrating factor ***(M1)***

$\int \frac{2x}{1+x^2}\mathrm{d}x = \ln(1+x^2)$ ***(M1)A1***

IF is $\mathrm{e}^{\ln(1+x^2)}$ ***(M1)A1***

$= 1 + x^2$ ***AG***

METHOD 2

multiply by the integrating factor

$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x^2(1+x^2)$ ***M1A1***

left hand side is equal to the derivative of $(1+x^2)y$

A3

[5 marks]

b. $(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = (1+x^2)x^2$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1+x^2)y\right] = x^2 + x^4$

$(1+x^2)y = \left(\int x^2 + x^4\mathrm{d}x =\right)\ \frac{x^3}{3} + \frac{x^5}{5}(+c)$ ***A1A1***

$y = \frac{1}{1+x^2}\left(\frac{x^3}{3} + \frac{x^5}{5} + c\right)$

$x = 0,\ y = 2 \Rightarrow c = 2$ ***M1A1***

$y = \frac{1}{1+x^2}\left(\frac{x^3}{3} + \frac{x^5}{5} + 2\right)$ ***A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the function $f(x) = \frac{1}{1+x^2},\ x \in \mathbb{R}$.

a. Illustrate graphically the inequality, $\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \int_0^1 f(x)\mathrm{d}x < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$. [3]

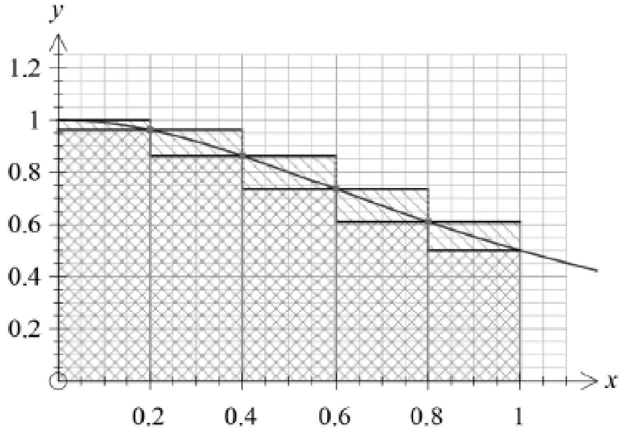
b. Use the inequality in part (a) to find a lower and upper bound for π. [5]

c. Show that $\sum_{r=0}^{n-1} (-1)^r x^{2r} = \frac{1+(-1)^{n-1}x^{2n}}{1+x^2}$. [2]

d. Hence show that $\pi = 4\left(\sum_{r=0}^{n-1} \frac{(-1)^r}{2r+1} - (-1)^{n-1}\int_0^1 \frac{x^{2n}}{1+x^2}\mathrm{d}x\right)$. [4]

Markscheme

a.



A1A1A1

A1 for upper rectangles, ***A1*** for lower rectangles, ***A1*** for curve in between with $0 \le x \le 1$

hence $\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \int_0^1 f(x)\mathrm{d}x < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$ ***AG***

[3 marks]

b. attempting to integrate from 0 to 1 ***(M1)***

$\int_0^1 f(x)\mathrm{d}x = [\arctan x]_0^1$

$= \frac{\pi}{4}$ ***A1***

attempt to evaluate either summation ***(M1)***

$\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \frac{\pi}{4} < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$

hence $\frac{4}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \pi < \frac{4}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$

so $2.93 < \pi < 3.33$ ***A1A1***

Note: Accept any answers that round to 2.9 and 3.3.

[5 marks]

c. **EITHER**

recognise $\sum_{r=0}^{n-1} (-1)^r x^{2r}$ as a geometric series with $r = -x^2$ ***M1***

sum of n terms is $\frac{1-(-x^2)^n}{1--x^2} = \frac{1+(-1)^{n-1}x^{2n}}{1+x^2}$ ***M1AG***

OR

$\sum_{r=0}^{n-1} (-1)^r (1+x^2)x^{2r} = (1+x^2)x^0 - (1+x^2)x^2 + (1+x^2)x^4 + \ldots$

$+(-1)^{n-1}(1+x^2)x^{2n-2}$ ***M1***

cancelling out middle terms ***M1***

$= 1 + (-1)^{n-1}x^{2n}$ ***AG***

[2 marks]

d. $\sum_{r=0}^{n-1} (-1)^r x^{2r} = \frac{1}{1+x^2} + (-1)^{n-1}\frac{x^{2n}}{1+x^2}$

integrating from 0 to 1 ***M1***

$\left[\sum_{r=0}^{n-1}(-1)^r \frac{x^{2r+1}}{2r+1}\right]_0^1 = \int_0^1 f(x)\mathrm{d}x + (-1)^{n-1}\int_0^1 \frac{x^{2n}}{1+x^2}\mathrm{d}x$ ***A1A1***

$\int_0^1 f(x)\mathrm{d}x = \frac{\pi}{4}$ ***A1***

so $\pi = 4\left(\sum_{r=0}^{n-1}\frac{(-1)^r}{2r+1} - (-1)^{n-1}\int_0^1 \frac{x^{2n}}{1+x^2}\mathrm{d}x\right)$ ***AG***

[4 marks]

Total [14 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

Use L'Hôpital's Rule to find $\lim\limits_{x\to 0}\frac{\mathrm{e}^x - 1 - x\cos x}{\sin^2 x}$.

Markscheme

apply l'Hôpital's Rule to a $0/0$ type limit

$\lim\limits_{x\to 0}\frac{\mathrm{e}^x-1-x\cos x}{\sin^2 x} = \lim\limits_{x\to 0}\frac{\mathrm{e}^x-\cos x+x\sin x}{2\sin x\cos x}$ ***M1A1***

noting this is also a $0/0$ type limit, apply l'Hôpital's Rule again ***(M1)***

obtain $\lim\limits_{x\to 0}\frac{\mathrm{e}^x+\sin x+x\cos x+\sin x}{2\cos 2x}$ ***A1***

substitution of $x = 0$ ***(M1)***

$= 0.5$ ***A1***

[6 marks]

Examiners report

The vast majority of candidates were familiar with L'Hôpitals rule and were also able to apply the technique twice as required by the problem. The errors that occurred were mostly due to difficulty in applying the differentiation rules correctly or errors in algebra. A small minority of candidates tried to use the quotient rule but it seemed that most candidates had a good understanding of L'Hôpital's rule and its application to finding a limit.

A function f is given by $f(x) = \int_0^x \ln(2+\sin t)\mathrm{d}t$.

a. Write down $f'(x)$. [1]

b. By differentiating $f(x^2)$, obtain an expression for the derivative of $\int_0^{x^2}\ln(2+\sin t)\mathrm{d}t$ with respect to x. [3]

c. Hence obtain an expression for the derivative of $\int_x^{x^2}\ln(2+\sin t)\mathrm{d}t$ with respect to x. [3]

Markscheme

a. $\ln(2+\sin x)$ ***A1***

Note: Do not accept $\ln(2+\sin t)$.

[1 mark]

b. attempt to use chain rule ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x^2)\right) = 2xf'(x^2)$ ***(A1)***

$= 2x\ln\left(2+\sin(x^2)\right)$ ***A1***

[3 marks]

c. $\int_x^{x^2} \ln(2+\sin t)\mathrm{d}t = \int_0^{x^2} \ln(2+\sin t)\mathrm{d}t - \int_0^x \ln(2+\sin t)\mathrm{d}t$ ***(M1)(A1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{x^2} \ln(2+\sin t)\mathrm{d}t\right) = 2x\ln\left(2+\sin(x^2)\right) - \ln(2+\sin x)$ ***A1***

[3 marks]

Examiners report

a. Many candidates answered this question well. Many others showed no knowledge of this part of the option; candidates that recognized the Fundamental Theorem of Calculus answered this question well. In general the scores were either very low or full marks.

b. Many candidates answered this question well. Many others showed no knowledge of this part of the option; candidates that recognized the Fundamental Theorem of Calculus answered this question well. In general the scores were either very low or full marks.

c. Many candidates answered this question well. Many others showed no knowledge of this part of the option; candidates that recognized the Fundamental Theorem of Calculus answered this question well. In general the scores were either very low or full marks.

The function f is defined by $f(x) = \mathrm{e}^x \sin x,\ x \in \mathbb{R}$.

The Maclaurin series is to be used to find an approximate value for $f(0.5)$.

a. By finding a suitable number of derivatives of f, determine the Maclaurin series for $f(x)$ as far as the term in x^3. [7]

b. Hence, or otherwise, determine the exact value of $\lim\limits_{x\to 0} \frac{\mathrm{e}^x \sin x - x - x^2}{x^3}$. [3]

c. (i) Use the Lagrange form of the error term to find an upper bound for the absolute value of the error in this approximation. [7]

(ii) Deduce from the Lagrange error term whether the approximation will be greater than or less than the actual value of $f(0.5)$.

Markscheme

a. attempt to use product rule ***(M1)***

$f'(x) = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x$ ***A1***

$f''(x) = 2\mathrm{e}^x \cos x$ ***A1***

$f''(x) = 2\mathrm{e}^x \cos x - 2\mathrm{e}^x \sin x$ ***A1***

$f(0) = 0,\ f'(0) = 1$

$f''(0) = 2,\ f'''(0) = 2$ ***(M1)***

$\mathrm{e}^x \sin x = x + x^2 + \frac{x^3}{3} + \ldots$ ***(M1)A1***

[7 marks]

b. **METHOD 1**

$\frac{\mathrm{e}^x \sin x - x - x^2}{x^3} = \frac{x + x^2 + \frac{x^3}{3} + \ldots - x - x^2}{x^3}$ ***M1A1***

$\to \frac{1}{3}$ as $x \to 0$ ***A1***

METHOD 2

$\lim_{x \to 0} \frac{\mathrm{e}^x \sin x - x - x^2}{x^3} = \lim_{x \to 0} \frac{\mathrm{e}^x \sin x + \mathrm{e}^x \cos x - 1 - 2x}{3x^2}$ ***A1***

$= \lim_{x \to 0} \frac{2\mathrm{e}^x \cos x - 2}{6x}$ ***A1***

$= \lim_{x \to 0} \frac{2\mathrm{e}^x \cos x - 2\mathrm{e}^x \sin x}{6} = \frac{1}{3}$ ***A1***

[3 marks]

c. (i) attempt to find 4^{th} derivative from the 3^{rd} derivative obtained in (a) ***M1***

$f''''(x) = -4\mathrm{e}^x \sin x$ ***A1***

Lagrange error term $= \frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}$ (where *c* lies between 0 and x)

$= -\frac{4\mathrm{e}^c \sin c \times 0.5^4}{4!}$ ***(M1)***

the maximum absolute value of this expression occurs when $c = 0.5$ ***(A1)***

Note: This ***A1*** is independent of previous ***M*** marks.

therefore

upper bound $= \frac{4\mathrm{e}^{0.5} \sin 0.5 \times 0.5^4}{4!}$ ***(M1)***

$= 0.00823$ ***A1***

(ii) the approximation is greater than the actual value because the Lagrange error term is negative ***R1***

[7 marks]

Examiners report

a. This part of the question was well answered by most candidates. In a few cases candidates failed to follow instructions and attempted to use known series; in a few cases mistakes in the determination of the derivatives prevented other candidates from achieving full marks; part (b) was also well answered using both the Maclaurin expansion or L'Hôpital rule; again in most cases that candidates failed to achieve full marks were due to mistakes in the determination of derivatives.

b. Part (a) of the question was well answered by most candidates. In a few cases candidates failed to follow instructions and attempted to use known series; in a few cases mistakes in the determination of the derivatives prevented other candidates from achieving full marks; part (b) was also well answered using both the Maclaurin expansion or L'Hôpital rule; again in most cases that candidates failed to achieve full marks were due to mistakes in the determination of derivatives.

c. Part (c) was poorly answered with few candidates showing familiarity with this part of the option. Most candidates quoted the formula and managed to find the 4^{th} derivative of f but then could not use it to obtain the required answer; in other cases candidates did obtain an answer but showed little understanding of its meaning when answering (c)(ii).

Consider the infinite series $S = \sum\limits_{n=0}^{\infty} u_n$ where $u_n = \int_{nx}^{(n+1)\pi} \frac{\sin t}{t}\mathrm{d}t$.

a. Explain why the series is alternating. [1]

b. (i) Use the substitution $T = t - \pi$ in the expression for u_{n+1} to show that $|u_{n+1}| < |u_n|$. [9]

(ii) Show that the series is convergent.

c. Show that $S < 1.65$. [4]

Markscheme

a. as t moves through the intervals $[0,\ \pi],\ [\pi,\ 2\pi],\ [2\pi,\ 3\pi],\ [3\pi,\ 4\pi]$, *etc*, the sign of $\sin t$, (and therefore the sign of the integral) alternates $+,\ -,\ +,\ -$, *etc*, so that the series is alternating ***R1***

Note: Award ***R1*** only if it includes a clear reason that justifies that the sign of the integrand alternates between – and + and this pattern is valid for all the terms.

The change of signs can be justified by a labelled graph of $y = \sin(x)$ or $y = \frac{\sin x}{x}$ that shows the intervals $[0,\ \pi],\ [\pi,\ 2\pi],\ [2\pi,\ 3\pi],\ \ldots$

[1 mark]

b. (i) $u_{n+1} = \int_{(n+1)\pi}^{(n+2)\pi} \frac{\sin t}{t}\mathrm{d}t$

(M1)

put $T = t - -\pi$ and $\mathrm{d}T = \mathrm{d}t$ ***(M1)***

the limits change to $n\pi,\ (n+1)\pi$

$|u_{n+1}| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin(T+\pi)|}{T+\pi}\mathrm{d}T$ (or equivalent) ***A1***

$|\sin(T+\pi)| = |\sin(T)|$ or $\sin(T+\pi) = -\sin(T)$ ***(M1)***

$= \int_{n\pi}^{(n+1)\pi} \frac{|\sin T|}{T+\pi}\mathrm{d}T$

$< \int_{n\pi}^{(n+1)\pi} \frac{|\sin T|}{T}\mathrm{d}T = |u_n|$ ***A1AG***

(ii) $|u_n| = \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t}\mathrm{d}t$

$< \int_{n\pi}^{(n+1)\pi} \frac{1}{t}\mathrm{d}t$ ***M1***

$= [\ln t]_{n\pi}^{(n+1)\pi}$ ***A1***

$= \ln\left(\frac{n+1}{n}\right)$ ***A1***

$\rightarrow \ln 1 = 0$ as $n \rightarrow \infty$

from part (i) $|u_n|$ is a decreasing sequence and since $\lim\limits_{n\to\infty} |u_n| = 0,$ ***R1***

the series is convergent ***AG***

[9 marks]

c. attempt to calculate the partial sums $\sum\limits_{i=0}^{n-1} u_i = \int_0^{n\pi} \frac{\sin t}{t} \mathrm{d}t$ ***(M1)***

the first partial sums are

n	$\sum\limits_{i=0}^{n-1} u_i$
1	1.85 (or 1.8519…)
2	1.42 (or 1.4181…)
3	1.67 (or 1.6747…)
4	1.49 (or 1.4921…)
5	1.63 (or 1.6339…)

two consecutive partial sums for $n \geqslant 4$ ***A1A1***

(*eg* $S_4 = 1.49$ and $S_5 = 1.63$ or $S_{100} = 1.567\ldots$ and $S_{101} = 1.573\ldots$)

Note: These answers must be given to a minimum of 3 significant figures.

the sum to infinity lies between any two consecutive partial sums,

eg between 1.49 and 1.63 ***R1***

so that $S < 1.65$ ***AG***

Note: Award ***A1A1R1*** to candidates who calculate at least two partial sums for only odd values of n and state that the upper bound is less than these values.

[4 marks]

Examiners report

a. Very few candidates presented a valid reason to justify the alternating nature of the series. In most cases candidates just reformulated the wording of the question by saying that it changed signs and completely ignored the interval over which the expression had to be integrated to obtain each term.

b. (i) Most candidates achieved 1 or 2 marks for attempting the given substitution; in most cases candidates failed to find the correct limits of integration for the new variable and then relate the expressions of the consecutive terms of the series. In part (ii) very few correct attempts were seen; in some cases candidates did recognize the conditions for the alternating series to be convergent but very few got close to establish that the limit of the general term was zero.

c. A few good attempts to use partial sums were seen although once again candidates showed difficulties in identifying what was needed to show the given answer. In most cases candidates just verified with GDC that in fact for high values of *n* the series was indeed less than the upper bound given but could not provide a valid argument that justified the given statement.

The curves $y = f(x)$ and $y = g(x)$ both pass through the point $(1,\ 0)$ and are defined by the differential equations $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x^2$ respectively.

a. Show that the tangent to the curve $y = f(x)$ at the point $(1,\ 0)$ is normal to the curve $y = g(x)$ at the point $(1,\ 0)$. [2]

b. Find $g(x)$. [6]

c. Use Euler's method with steps of 0.2 to estimate $f(2)$ to 5 decimal places. [5]

d. Explain why $y = f(x)$ cannot cross the isocline $x - y^2 = 0$, for $x > 1$. [3]

e. (i) Sketch the isoclines $x - y^2 = -2,\ 0,\ 1$. [4]

(ii) On the same set of axes, sketch the graph of f.

Markscheme

a. gradient of f at $(1,\ 0)$ is $1 - 0^2 = 1$ and the gradient of g at $(1,\ 0)$ is $0 - 1^2 = -1$ ***A1***

so gradient of normal is 1 ***A1***

= Gradient of the tangent of f at $(1,\ 0)$ ***AG***

[2 marks]

b. $\frac{\mathrm{d}y}{\mathrm{d}x} - y = -x^2$

integrating factor is $\mathrm{e}^{\int -1\mathrm{d}x} = \mathrm{e}^{-x}$ ***M1***

$y\mathrm{e}^{-x} = \int -x^2\mathrm{e}^{-x}\mathrm{d}x$ ***A1***

$= x^2\mathrm{e}^{-x} - \int 2x\mathrm{e}^{-x}\mathrm{d}x$ ***M1***

$= x^2\mathrm{e}^{-x} + 2x\mathrm{e}^{-x} - \int 2\mathrm{e}^{-x}\mathrm{d}x$

$= x^2\mathrm{e}^{-x} + 2x\mathrm{e}^{-x} + 2\mathrm{e}^{-x} + c$ ***A1***

Note: Condone missing $+c$ at this stage.

$\Rightarrow g(x) = x^2 + 2x + 2 + c\mathrm{e}^x$

$g(1) = 0 \Rightarrow c = -\frac{5}{\mathrm{e}}$ ***M1***

$\Rightarrow g(x) = x^2 + 2x + 2 - 5\mathrm{e}^{x-1}$ ***A1***

[6 marks]

c. use of $y_{n+1} = y_n + hf'(x_n,\ y_n)$ ***(M1)***

$x_0 = 1,\ y_0 = 0$

$x_1 = 1.2,\ y_1 = 0.2$ ***A1***

$x_2 = 1.4,\ y_2 = 0.432$ ***(M1)(A1)***

$x_3 = 1.6,\ y_3 = 0.67467\ldots$

$x_4 = 1.8,\ y_4 = 0.90363\ldots$

$x_5 = 2,\ y_5 = 1.1003255\ldots$

answer $= 1.10033$ ***A1*** ***N3***

Note: Award ***A0*** or ***N1*** if 1.10 given as answer.

[5 marks]

d. at the point $(1,\ 0)$, the gradient of f is positive so the graph of f passes into the first quadrant for $x > 1$

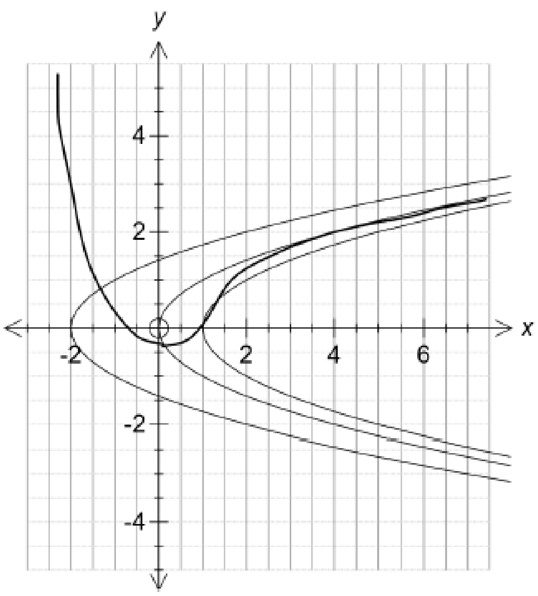
in the first quadrant below the curve $x - y^2 = 0$ the gradient of f is positive ***R1***

the curve $x - y^2 = 0$ has positive gradient in the first quadrant ***R1***

if f were to reach $x - y^2 = 0$ it would have gradient of zero, and therefore would not cross ***R1***

[3 marks]

e. (i) and (ii)



A4

Note: Award ***A1*** for 3 correct isoclines.

Award ***A1*** for f not reaching $x - y^2 = 0$.

Award ***A1*** for turning point of f on $x - y^2 = 0$.

Award ***A1*** for negative gradient to the left of the turning point.

Note: Award ***A1*** for correct shape and position if curve drawn without any isoclines.

[4 marks]

Total [20 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]
e. [N/A]

Use the integral test to determine whether the infinite series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ is convergent or divergent.

Markscheme

consider $I = \int\limits_2^N \frac{\mathrm{d}x}{x\sqrt{\ln x}}$ ***M1A1***

Note: Do not award ***A1*** if n is used as the variable or if lower limit equal to 1, but some subsequent ***A*** marks can still be awarded. Allow ∞ as upper limit.

let $y = \ln x$ ***(M1)***

$\mathrm{d}y = \frac{\mathrm{d}x}{x}$, ***(A1)***

$[2,\ N] \Rightarrow [\ln 2,\ \ln N]$

$I = \int\limits_{\ln\ 2}^{\ln\ N} \frac{\mathrm{d}y}{\sqrt{y}}$ ***(A1)***

Note: Condone absence of limits, or wrong limits.

$= \left[2\sqrt{y}\right]_{\ln 2}^{\ln N}$ ***A1***

Note: ***A1*** is for the correct integral, irrespective of the limits used. Accept correct use of integration by parts.

$= 2\sqrt{\ln N} - 2\sqrt{\ln 2}$ ***(M1)***

Note: ***M1*** is for substituting their limits into their integral and subtracting.

$\to \infty$ as $N \to \infty$ ***A1***

Notes: Allow "$= \infty$", "limit does not exist", "diverges" or equivalent.

Do not award if wrong limits substituted into the integral but allow N or ∞ as an upper limit in place of $\ln N$.

(by the integral test) the series is divergent (because the integral is divergent) ***A1***

Notes: Do not award this mark if ∞ used as upper limit throughout.

[9 marks]

Examiners report

[N/A]

Let $S = \sum\limits_{n=1}^{\infty} \frac{(x-3)^n}{n^2+2}$.

a. Use the limit comparison test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ is convergent. [3]

b. Find the interval of convergence for S. [9]

Markscheme

a. $\lim_{n\to\infty} \frac{\frac{1}{n^2+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+2} = \left(\lim_{n\to\infty} \left(1 - \frac{2}{n^2+2}\right)\right)$ ***M1***

$= 1$ ***A1***

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (a p-series with $p = 2$) ***R1***

by limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ also converges ***AG***

Notes: The ***R1*** is independent of the ***A1***.

[3 marks]

b. applying the ratio test $\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{(n+1)^2+2} \times \frac{n^2+2}{(x-3)^n} \right|$ ***M1A1***

$= |x-3| \ \left(\text{as } \lim_{n\to\infty} \frac{(n^2+2)}{(n+1)^2+2} = 1\right)$ ***A1***

converges if $|x-3| < 1$ (converges for $2 < x < 4$) ***M1***

considering endpoints $x = 2$ and $x = 4$ ***M1***

when $x = 4$, series is $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$, convergent from (a) ***A1***

when $x = 2$, series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$ ***A1***

EITHER

$\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ is convergent therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2}$ is (absolutely) convergent ***R1***

OR

$\frac{1}{n^2+2}$ is a decreasing sequence and $\lim_{n\to\infty} \frac{1}{n^2+2} = 0$ so series converges by the alternating series test ***R1***

THEN

interval of convergence is $2 \leqslant x \leqslant 4$ ***A1***

Note: The final ***A1*** is dependent on previous ***A1***s – *ie*, considering correct series when $x = 2$ and $x = 4$ and on the final ***R1***.

[9 marks]

Examiners report

[N/A]

b. [N/A]

The mean value theorem states that if f is a continuous function on $[a,\ b]$ and differentiable on $]a,\ b[$ then $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some $c \in]a,\ b[$.

The function g, defined by $g(x) = x\cos\left(\sqrt{x}\right)$, satisfies the conditions of the mean value theorem on the interval $[0,\ 5\pi]$.

a. For $a = 0$ and $b = 5\pi$, use the mean value theorem to find all possible values of c for the function g. [6]

b. Sketch the graph of $y = g(x)$ on the interval $[0,\ 5\pi]$ and hence illustrate the mean value theorem for the function g. [4]

Markscheme

a. $\frac{g(5\pi)-g(0)}{5\pi-0} = -0.6809\ldots\ \left(= \cos\sqrt{5\pi}\right)$ (gradient of chord) ***(A1)***

$g'(x) = \cos\left(\sqrt{x}\right) - \frac{\sqrt{x}\sin\left(\sqrt{x}\right)}{2}$ (or equivalent) ***(M1)(A1)***

Note: Award ***M1*** to candidates who attempt to use the product and chain rules.

attempting to solve $\cos\left(\sqrt{c}\right) - \frac{\sqrt{c}\sin\left(\sqrt{c}\right)}{2} = -0.6809\ldots$ for c ***(M1)***

Notes: Award ***M1*** to candidates who attempt to solve their $g'(c) =$ gradient of chord.

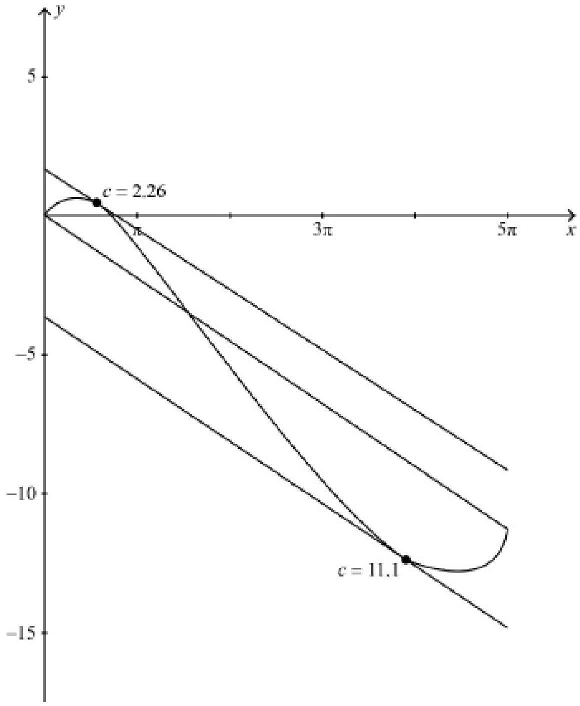
Do not award ***M1*** to candidates who just attempt to rearrange their equation.

$c = 2.26,\ 11.1$ ***A1A1***

Note: Condone candidates working in terms of x.

[6 marks]

b.



correct graph: 2 turning points close to the endpoints, endpoints indicated and correct endpoint behaviour ***A1***

Notes: Endpoint coordinates are not required. Candidates do not need to indicate axes scales.

correct chord ***A1***

tangents drawn at their values of c which are approximately parallel to the chord ***A1A1***

Notes: Award ***A1A0A1A0*** to candidates who draw a correct graph, do not draw a chord but draw 2 tangents at their values of c. Condone the absence of their $c-$ values stated on their sketch. However do not award marks for tangents if no $c-$ values were found in (a).

[4 marks]

Examiners report

a. [N/A]

b. [N/A]

Consider the infinite series $\sum\limits_{n=1}^{\infty} \frac{(n-1)x^n}{n^2 \times 2^n}$.

a. Find the radius of convergence. [4]

b. Find the interval of convergence. [9]

Markscheme

a. using the ratio test, $\frac{u_{n+1}}{u_n} = \frac{nx^{n+1}}{(n+1)^2 2^{n+1}} \times \frac{n^2 2^n}{(n-1)x^n}$ ***M1***

$= \frac{n^3}{(n+1)^2(n-1)} \times \frac{x}{2}$ ***A1***

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{x}{2}$ ***A1***

the radius of convergence *R* satisfies

$\frac{R}{2} = 1$ so $R = 2$ ***A1***

[4 marks]

b. considering $x = 2$ for which the series is

$\sum_{n=1}^{\infty} \frac{(n-1)}{n^2}$

using the limit comparison test with the harmonic series ***M1***

$\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges

consider

$\lim_{n\to\infty} \frac{u_n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n-1}{n} = 1$ ***A1***

the series is therefore divergent for $x = 2$ ***A1***

when $x = -2$, the series is

$\sum_{n=1}^{\infty} \frac{(n-1)}{n^2} \times (-1)^n$

this is an alternating series in which the n^{th} term tends to 0 as $n \to \infty$ ***A1***

consider $f(x) = \frac{x-1}{x^2}$ ***M1***

$f'(x) = \frac{2-x}{x^3}$ ***A1***

this is negative for $x > 2$ so the sequence $\{|u_n|\}$ is eventually decreasing ***R1***

the series therefore converges when $x = -2$ by the alternating series test ***R1***

the interval of convergence is therefore [–2, 2[***A1***

[9 marks]

Examiners report

a. [N/A]
b. [N/A]

Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{1+x}$, where $x > -1$ and $y = 1$ when $x = 0$.

a. Use Euler's method, with a step length of 0.1, to find an approximate value of *y* when $x = 0.5$. [7]

b. (i) Show that $\frac{d^2y}{dx^2} = \frac{2y^3-y^2}{(1+x)^2}$. [8]

(ii) Hence find the Maclaurin series for *y*, up to and including the term in x^2 .

c. (i) Solve the differential equation. [6]

(ii) Find the value of *a* for which $y \to \infty$ as $x \to a$.

Markscheme

a. attempt the first step of

$y_{n+1} = y_n + (0.1)f(x_n,\, y_n)$ with $y_0 = 1,\ x_0 = 0$ ***(M1)***
$y_1 = 1.1$ ***A1***
$y_2 = 1.1 + (0.1)\frac{1.1^2}{1.1} = 1.21$ ***(M1)A1***
$y_3 = 1.332(0)$ ***(A1)***
$y_4 = 1.4685$ ***(A1)***
$y_5 = 1.62$ ***A1***
[7 marks]

b. (i) recognition of both quotient rule and implicit differentiation ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{(1+x)2y\frac{\mathrm{d}y}{\mathrm{d}x} - y^2\times 1}{(1+x)^2}$ ***A1A1***

Note: Award ***A1*** for first term in numerator, ***A1*** for everything else correct.

$= \frac{(1+x)2y\frac{y^2}{1+x} - y^2\times 1}{(1+x)^2}$ ***M1A1***

$= \frac{2y^3 - y^2}{(1+x)^2}$ ***AG***

(ii) attempt to use $y = y(0) + x\frac{\mathrm{d}y}{\mathrm{d}x}(0) + \frac{x^2}{2!}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}(0) + \ldots$ ***(M1)***

$= 1 + x + \frac{x^2}{2}$ ***A1A1***

Note: Award ***A1*** for correct evaluation of $y(0),\ \frac{dy}{dx}(0),\ \frac{d^2y}{dx^2}(0)$, ***A1*** for correct series.

[8 marks]

c. (i) separating the variables $\int \frac{1}{y^2}\mathrm{d}y = \int \frac{1}{1+x}\mathrm{d}x$ ***M1***

obtain $-\frac{1}{y} = \ln(1+x) + (c)$ ***A1***
impose initial condition $-1 = \ln 1 + c$ ***M1***
obtain $y = \frac{1}{1-\ln(1+x)}$ ***A1***

(ii) $y \to \infty$ if $\ln(1+x) \to 1$, so $a = \mathrm{e} - 1$ ***(M1)A1***
Note: To award ***A1*** must see either $x \to e - 1$ or $a = e - 1$. Do not accept $x = e - 1$.

[6 marks]

Examiners report

a. Most candidates had a good knowledge of Euler’s method and were confident in applying it to the differential equation in part (a). A few candidates who knew the Euler’s method completed one iteration too many to arrive at an incorrect answer but this was rare. Nearly all candidates who applied the correct technique in part (a) correctly calculated the answer. Most candidates were able to attempt part (b) but some lost marks due to a lack of rigour by not clearly showing the implicit differentiation in the first line of working. Part (c) was reasonably well attempted by many candidates and many could solve the integrals although some did not find the arbitrary constant meaning that it was not possible to solve (ii) of the part (c).

b. Most candidates had a good knowledge of Euler’s method and were confident in applying it to the differential equation in part (a). A few candidates who knew the Euler’s method completed one iteration too many to arrive at an incorrect answer but this was rare. Nearly all candidates who applied the correct technique in part (a) correctly calculated the answer. Most candidates were able to attempt part (b) but some lost marks due to a lack of rigour by not clearly showing the implicit differentiation in the first line of working. Part (c) was reasonably well attempted by many candidates and many could solve the integrals although some did not find the arbitrary constant meaning that it was not possible to solve (ii) of the part (c).

c. Most candidates had a good knowledge of Euler's method and were confident in applying it to the differential equation in part (a). A few candidates who knew the Euler's method completed one iteration too many to arrive at an incorrect answer but this was rare. Nearly all candidates who applied the correct technique in part (a) correctly calculated the answer. Most candidates were able to attempt part (b) but some lost marks due to a lack of rigour by not clearly showing the implicit differentiation in the first line of working. Part (c) was reasonably well attempted by many candidates and many could solve the integrals although some did not find the arbitrary constant meaning that it was not possible to solve (ii) of the part (c).

(a) Show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \cos^2 y,$$

given that $y = \frac{\pi}{4}$ when $x = \pi$, is $y = \arctan(1 + \sin x)$.

(b) Determine the value of the constant a for which the following limit exists

$$\lim_{x \to \frac{\pi}{2}} \frac{\arctan(1 + \sin x) - a}{\left(x - \frac{\pi}{2}\right)^2}$$

and evaluate that limit.

Markscheme

(a) this separable equation has general solution

$\int \sec^2 y \mathrm{d}y = \int \cos x \mathrm{d}x$ ***(M1)(A1)***

$\tan y = \sin x + c$ ***A1***

the condition gives

$\tan \frac{\pi}{4} = \sin \pi + c \Rightarrow c = 1$ ***M1***

the solution is $\tan y = 1 + \sin x$ ***A1***

$y = \arctan(1 + \sin x)$ ***AG***

[5 marks]

(b) the limit cannot exist unless $a = \arctan\left(1 + \sin \frac{\pi}{2}\right) = \arctan 2$ ***R1A1***

in that case the limit can be evaluated using l'Hopital's rule (twice) limit is

$\lim_{x \to \frac{\pi}{2}} \frac{(\arctan(1+\sin x))'}{2\left(x - \frac{\pi}{2}\right)} = \lim_{x \to \frac{\pi}{2}} \frac{y'}{2\left(x - \frac{\pi}{2}\right)}$ ***M1A1***

where y is the solution of the differential equation

the numerator has zero limit (from the factor $\cos x$ in the differential equation) ***R1***

so required limit is

$\lim_{x \to \frac{\pi}{2}} \frac{y''}{2}$ ***M1A1***

finally,

$y'' = -\sin x \cos^2 y - 2\cos x \cos y \sin y \times y'(x)$ ***M1A1***

since $\cos y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{5}}$ ***A1***

$y'' = -\frac{1}{5}$ at $x = \frac{\pi}{2}$ ***A1***

the required limit is $-\frac{1}{10}$ ***A1***

[12 marks]

Total [17 marks]

Examiners report

Many candidates successfully obtained the displayed solution of the differential equation in part(a). Few complete solutions to part(b) were seen which used the result in part(a). The problem can, however, be solved by direct differentiation although this is algebraically more complicated. Some successful solutions using this method were seen.

---

a. Find the radius of convergence of the infinite series [7]

$$\frac{1}{2}x + \frac{1 \times 3}{2 \times 5}x^2 + \frac{1 \times 3 \times 5}{2 \times 5 \times 8}x^3 + \frac{1 \times 3 \times 5 \times 7}{2 \times 5 \times 8 \times 11}x^4 + \dots .$$

b. Determine whether the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n} + n\pi\right)$ is convergent or divergent. [8]

Markscheme

a. the *n*th term is

$u_n = \frac{1 \times 3 \times 5 \ldots (2n-1)}{2 \times 5 \times 8 \ldots (3n-1)}x^n$ ***M1A1***

(using the ratio test to test for absolute convergence)

$\frac{|u_{n+1}|}{|u_n|} = \frac{(2n+1)}{(3n+2)}|x|$ ***M1A1***

$\lim_{n \to \infty} \frac{|u_{n+1}|}{|u_n|} = \frac{2|x|}{3}$ ***A1***

let *R* denote the radius of convergence

then $\frac{2R}{3} = 1$ so $r = \frac{3}{2}$ ***M1A1***

Note: Do not penalise the absence of absolute value signs.

[7 marks]

b. using the compound angle formula or a graphical method the series can be written in the form ***(M1)***

$\sum_{n=1}^{\infty} u_n$ where $u_n = (-1)^n \sin\left(\frac{1}{n}\right)$ ***A2*** since $\frac{1}{n} < \frac{\pi}{2}$ *i.e.* an angle in the first quadrant, ***R1***

it is an alternating series ***R1***

$u_n \to 0$ as $n \to \infty$ ***R1***

and $|u_{n+1}| < |u_n|$ ***R1***

it follows that the series is convergent ***R1***

[8 marks]

Examiners report

a. Solutions to this question were generally disappointing. In (a), many candidates were unable even to find an expression for the *n*th term so that they could not apply the ratio test.

b. Solutions to this question were generally disappointing. In (b), few candidates were able to rewrite the *n*th term in the form $\sum (-1)^n \sin\left(\frac{1}{n}\right)$ so that most candidates failed to realise that the series was alternating.

Consider the functions f and g given by $f(x) = \frac{e^x + e^{-x}}{2}$ and $g(x) = \frac{e^x - e^{-x}}{2}$.

a. Show that $f'(x) = g(x)$ and $g'(x) = f(x)$. [2]

b. Find the first three non-zero terms in the Maclaurin expansion of $f(x)$. [5]

c. Hence find the value of $\lim\limits_{x \to 0} \frac{1 - f(x)}{x^2}$. [3]

d. Find the value of the improper integral $\int_0^\infty \frac{g(x)}{[f(x)]^2} dx$. [6]

Markscheme

a. any correct step before the given answer ***A1AG***

eg, $f'(x) = \frac{(e^x)' + (e^{-x})'}{2} = \frac{e^x - e^{-x}}{2} = g(x)$

any correct step before the given answer ***A1AG***

eg, $g'(x) = \frac{(e^x)' - (e^{-x})'}{2} = \frac{e^x + e^{-x}}{2} = f(x)$

[2 marks]

b. **METHOD 1**

statement and attempted use of the general Maclaurin expansion formula ***(M1)***

$f(0) = 1;\ g(0) = 0$ (or equivalent in terms of derivative values) ***A1A1***

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$ or $f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$ ***A1A1***

METHOD 2

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ ***A1***

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ ***A1***

adding and dividing by 2 ***M1***

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}$ or $f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$ ***A1A1***

Notes: Accept 1, $\frac{x^2}{2}$ and $\frac{x^4}{24}$ or 1, $\frac{x^2}{2!}$ and $\frac{x^4}{4!}$.

Award ***A1*** if two correct terms are seen.

[5 marks]

c. **METHOD 1**

attempted use of the Maclaurin expansion from (b) ***M1***

$\lim\limits_{x \to 0} \frac{1-f(x)}{x^2} = \lim\limits_{x \to 0} \frac{1-\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\ldots\right)}{x^2}$

$\lim\limits_{x \to 0}\left(-\frac{1}{2} - \frac{x^2}{24} - \ldots\right)$ ***A1***

$= -\frac{1}{2}$ ***A1***

METHOD 2

attempted use of L'Hôpital and result from (a) ***M1***

$\lim\limits_{x \to 0} \frac{1-f(x)}{x^2} = \lim\limits_{x \to 0} \frac{-g(x)}{2x}$

$\lim\limits_{x \to 0} \frac{-f(x)}{2}$ ***A1***

$= -\frac{1}{2}$ ***A1***

[3 marks]

d. **METHOD 1**

use of the substitution $u = f(x)$ and $(\mathrm{d}u = g(x)\mathrm{d}x)$ ***(M1)(A1)***

attempt to integrate $\int_1^\infty \frac{\mathrm{d}u}{u^2}$ ***(M1)***

obtain $\left[-\frac{1}{u}\right]_1^\infty$ or $\left[-\frac{1}{f(x)}\right]_0^\infty$ ***A1***

recognition of an improper integral by use of a limit or statement saying the integral converges ***R1***

obtain 1 ***A1*** ***N0***

METHOD 2

$\int_0^\infty \frac{\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2}}{\left(\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}\right)^2}\mathrm{d}x = \int_0^\infty \frac{2(\mathrm{e}^x-\mathrm{e}^{-x})}{(\mathrm{e}^x+\mathrm{e}^{-x})^2}\mathrm{d}x$ ***(M1)***

use of the substitution $u = \mathrm{e}^x + \mathrm{e}^{-x}$ and $(\mathrm{d}u = \mathrm{e}^x - \mathrm{e}^{-x}\mathrm{d}x)$ ***(M1)***

attempt to integrate $\int_2^\infty \frac{2\mathrm{d}u}{u^2}$ ***(M1)***

obtain $\left[-\frac{2}{u}\right]_2^\infty$ ***A1***

recognition of an improper integral by use of a limit or statement saying the integral converges ***R1***

obtain 1 ***A1*** ***N0***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

A differential equation is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$, where $x > 0$ and $y > 0$.

a. Solve this differential equation by separating the variables, giving your answer in the form $y = f(x)$. [3]

b. Solve the same differential equation by using the standard homogeneous substitution $y = vx$. [4]

c. Solve the same differential equation by the use of an integrating factor. [5]

d. If $y = 20$ when $x = 2$, find y when $x = 5$. [1]

Markscheme

a. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} \Rightarrow \int \frac{1}{y}\mathrm{d}y = \int \frac{1}{x}\mathrm{d}x$ ***M1***

$\Rightarrow \ln y = \ln x + c$ ***A1***

$\Rightarrow \ln y = \ln x + \ln k = \ln kx$

$\Rightarrow y = kx$ ***A1***

[3 marks]

b. $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

so $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v$ ***M1***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 0 \quad (\text{as } x \neq 0)$ ***R1***

$\Rightarrow v = k$

$\Rightarrow \frac{y}{x} = k \quad (\Rightarrow y = kx)$ ***A1***

[4 marks]

c. $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{-1}{x}\right)y = 0$ ***(M1)***

$\mathrm{IF} = \mathrm{e}^{\int \frac{-1}{x}\mathrm{d}x} = \mathrm{e}^{-\ln x} = \frac{1}{x}$ ***M1A1***

$x^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} - x^{-2}y = 0$

$\Rightarrow \frac{\mathrm{d}[x^{-1}y]}{\mathrm{d}x} = 0$ ***(M1)***

$\Rightarrow x^{-1}y = k \quad (\Rightarrow y = kx)$ ***A1***

[5 marks]

d. $20 = 2k \Rightarrow k = 10$ so $y(5) = 10 \times 5 = 50$ ***A1***

[1 mark]

Examiners report

a. This question allowed candidates to demonstrate a range of skills in solving differential equations. Generally this was well done with candidates making mistakes in algebra rather than the techniques themselves. For example a common error in part (a) was to go from $\ln y = \ln x + c$ to $y = x + c$

b. This question allowed candidates to demonstrate a range of skills in solving differential equations. Generally this was well done with candidates making mistakes in algebra rather than the techniques themselves. For example a common error in part (a) was to go from $\ln y = \ln x + c$ to $y = x + c$

c. This question allowed candidates to demonstrate a range of skills in solving differential equations. Generally this was well done with candidates making mistakes in algebra rather than the techniques themselves. For example a common error in part (a) was to go from $\ln y = \ln x + c$ to $y = x + c$

d. This question allowed candidates to demonstrate a range of skills in solving differential equations. Generally this was well done with candidates making mistakes in algebra rather than the techniques themselves. For example a common error in part (a) was to go from $\ln y = \ln x + c$ to $y = x + c$

The function f is defined by

$$f(x) = \begin{cases} |x-2|+1 & x < 2 \\ ax^2 + bx & x \geqslant 2 \end{cases}$$

where a and b are real constants

Given that both f and its derivative are continuous at $x = 2$, find the value of a and the value of b.

Markscheme

considering continuity at $x = 2$

$\lim\limits_{x \to 2^-} f(x) = 1$ and $\lim\limits_{x \to 2^+} f(x) = 4a + 2b$ ***(M1)***

$4a + 2b = 1$ ***A1***

considering differentiability at $x = 2$

$f'(x) = \begin{cases} -1 & x < 2 \\ 2ax + b & x \geqslant 2 \end{cases}$ ***(M1)***

$\lim\limits_{x \to 2^-} f'(x) = -1$ and $\lim\limits_{x \to 2^+} f'(x) = 4a + b$ ***(M1)***

Note: The above ***M1*** is for attempting to find the left and right limit of their derived piecewise function at $x = 2$.

$4a + b = -1$ ***A1***

$a = -\frac{3}{4}$ and $b = 2$ ***A1***

[6 marks]

Examiners report

[N/A]

---

a. The mean value theorem states that if f is a continuous function on $[a,\ b]$ and differentiable on $]a,\ b[$ then $f'(c) = \frac{f(b)-f(a)}{b-a}$ for some $c \in]a,\ b[$. [7]

(i) Find the two possible values of c for the function defined by $f(x) = x^3 + 3x^2 - 2$ on the interval $[-3,\ 1]$.

(ii) Illustrate this result graphically.

b. (i) The function f is continuous on $[a,\ b]$, differentiable on $]a,\ b[$ and $f'(x) = 0$ for all $x \in]a,\ b[$. Show that $f(x)$ is constant on $[a,\ b]$. [9]

(ii) Hence, prove that for $x \in [0,\ 1],\ 2\arccos x + \arccos(1 - 2x^2) = \pi$.

Markscheme

a. (i) $f'(x) = 3x^2 + 6x$ ***A1***

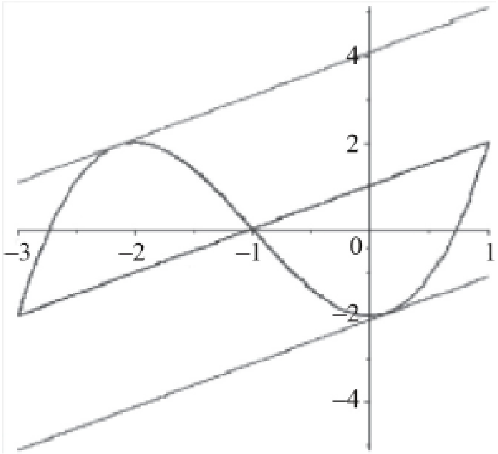
gradient of chord $= 1$ ***A1***

$3c^2 + 6c = 1$

$c = \frac{-3 \pm 2\sqrt{3}}{3}$ $(= -2.15,\ 0.155)$ ***A1A1***

Note: Accept any answers that round to the correct 2sf answers $(-2.2,\ 0.15)$.

(ii)



award ***A1*** for correct shape and clear indication of correct domain, ***A1*** for chord (from $x = -3$ to $x = 1$) and ***A1*** for two tangents drawn at their values of c ***A1A1A1***

[7 marks]

b. (i) **METHOD 1**

(if a theorem is true for the interval $[a,\ b]$, it is also true for any interval $[x_1,\ x_2]$ which belongs to $[a,\ b]$)

suppose $x_1,\ x_2 \in [a,\ b]$ ***M1***

by the MVT, there exists c such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$ ***M1A1***

hence $f(x_1) = f(x_2)$ ***R1***

as $x_1,\ x_2$ are arbitrarily chosen, $f(x)$ is constant on $[a,\ b]$

Note: If the above is expressed in terms of a and b award ***M0M1A0R0***.

METHOD 2

(if a theorem is true for the interval $[a,\ b]$, it is also true for any interval $[x_1,\ x_2]$ which belongs to $[a,\ b]$)

suppose $x \in [a,\ b]$ ***M1***

by the MVT, there exists c such that $f'(c) = \frac{f(x)-f(a)}{x-a} = 0$ ***M1A1***

hence $f(x) = f(a) =$ constant ***R1***

(ii) attempt to differentiate $(x) = 2\arccos x + \arccos(1-2x^2)$ ***M1***

$-2\frac{1}{\sqrt{1-x^2}} - \frac{-4x}{\sqrt{1-(1-2x^2)^2}}$ ***A1A1***

$= -2\frac{1}{\sqrt{1-x^2}} + \frac{4x}{\sqrt{4x^2-4x^4}} = 0$ ***A1***

Note: Only award ***A1*** for 0 if a correct attempt to simplify the denominator is also seen.

$f(x) = f(0) = 2 \times \frac{\pi}{2} + 0 = \pi$ ***A1AG***

Note: This ***A1*** is not dependent on previous marks.

Note: Allow any value of $x \in [0,\ 1]$.

[9 marks]

Total [16 marks]

Examiners report

a. (i) This was well done by most candidates.

(ii) This was generally poorly done, with many candidates failing to draw the curve correctly as they did not appreciate the importance of the given domain. Another common error was to draw the graph of the derivative rather than the function.

b. (i) This was very poorly done. A lot of the arguments seemed to be stating what was being required to be proved, eg 'because the derivative is equal to 0 the line is flat'. Most candidates did not realise the importance of testing a point inside the interval, so the most common solutions seen involved the Mean Value Theorem applied to the end points. In addition there was some confusion between the Mean Value Theorem and Rolle's Theorem.

(ii) It was pleasing that so many candidates spotted the link with the previous part of the question. The most common error after this point was to differentiate incorrectly. Candidates should be aware this is a 'prove' question, and so it was not sufficient simply to state, for example, $f(0) = \pi$.

Use l'Hôpital's rule to determine the value of

$$\lim_{x\to 0}\frac{\sin^2 x}{x\ln(1+x)}.$$

Markscheme

attempt to use l'Hôpital's rule, ***M1***

$\text{limit} = \lim_{x\to 0}\frac{2\sin x\cos x}{\ln(1+x)+\frac{x}{1+x}}$ or $\frac{\sin 2x}{\ln(1+x)+\frac{x}{1+x}}$ ***A1A1***

Note: Award ***A1*** for numerator ***A1*** for denominator.

this gives 0/0 so use the rule again ***(M1)***

$= \lim_{x\to 0}\frac{2\cos^2 x-2\sin^2 x}{\frac{1}{1+x}+\frac{1+x-x}{(1+x)^2}}$ or $\frac{2\cos 2x}{\frac{2+x}{(1+x)^2}}$ ***A1A1***

Note: Award ***A1*** for numerator ***A1*** for denominator.

$= 1$ ***A1***

Note: This ***A1*** is dependent on all previous marks being awarded, except when the first application of L'Hopital's does not lead to 0/0, when it should be awarded for the correct limit of their derived function.

[7 marks]

Examiners report

[N/A]

Solve the differential equation

$$(x-1)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=(x-1)\mathrm{e}^{-x}$$

given that $y = 1$ when $x = 0$. Give your answer in the form $y=f(x)$.

Markscheme

writing the differential equation in standard form gives

$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{x-1}y=\mathrm{e}^{-x}$ ***M1***

$\int\frac{x}{x-1}\mathrm{d}x=\int\left(1+\frac{1}{x-1}\right)\mathrm{d}x=x+\ln(x-1)$ ***M1A1***

hence integrating factor is $\mathrm{e}^{x+\ln(x-1)}=(x-1)\mathrm{e}^{x}$ ***M1A1***

hence, $(x-1)\mathrm{e}^{x}\frac{\mathrm{d}y}{\mathrm{d}x}+x\mathrm{e}^{x}y=x-1$ ***(A1)***

$\Rightarrow\frac{\mathrm{d}[(x-1)\mathrm{e}^{x}y]}{\mathrm{d}x}=x-1$ ***(A1)***

$\Rightarrow(x-1)\mathrm{e}^{x}y=\int(x-1)\mathrm{d}x$ ***A1***

$\Rightarrow(x-1)\mathrm{e}^{x}y=\frac{x^2}{2}-x+c$ ***A1***

substituting (0, 1), $c = -1$ ***(M1)A1***

$\Rightarrow(x-1)\mathrm{e}^{x}y=\frac{x^2-2x-2}{2}$ ***(A1)***

hence, $y=\frac{x^2-2x-2}{2(x-1)\mathrm{e}^{x}}$ (or equivalent) ***A1***

[13 marks]

Examiners report

Apart from some candidates who thought the differential equation was homogenous, the others were usually able to make a good start, and found it quite straightforward. Some made errors after identifying the correct integrating factor, and so lost accuracy marks.

---

a. Prove by induction that $n!>3^n$, for $n\geq 7,\ n\in\mathbb{Z}$. [5]

b. Hence use the comparison test to prove that the series $\sum\limits_{r=1}^{\infty}\frac{2^r}{r!}$ converges. [6]

Markscheme

a. if $n=7$ then $7!>3^7$ ***A1***

so true for $n = 7$

assume true for $n = k$ ***M1***

so $k! > 3^k$

consider $n = k + 1$

$(k+1)! = (k+1)k!$ ***M1***

$> (k+1)3^k$

$> 3.3k \quad (\text{as } k > 6)$ ***A1***

$= 3^{k+1}$

hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 7$, so true for all $n \geq 7$. ***R1***

Note: Do not award the ***R1*** if the two ***M*** marks have not been awarded.

[5 marks]

b. consider the series $\sum_{r=7}^{\infty} a_r$, where $a_r = \frac{2^r}{r!}$ ***R1***

Note: Award the ***R1*** for starting at $r = 7$

compare to the series $\sum_{r=7}^{\infty} b_r$ where $b_r = \frac{2^r}{3^r}$ ***M1***

$\sum_{r=7}^{\infty} b_r$ is an infinite Geometric Series with $r = \frac{2}{3}$ and hence converges ***A1***

Note: Award the ***A1*** even if series starts at $r = 1$.

as $r! > 3^r$ so $(0 <)a_r < b_r$ for all $r \geq 7$ ***M1R1***

as $\sum_{r=7}^{\infty} b_r$ converges and $a_r < b_r$ so $\sum_{r=7}^{\infty} a_r$ must converge

Note: Award the ***A1*** even if series starts at $r = 1$.

as $\sum_{r=1}^{6} a_r$ is finite, so $\sum_{r=1}^{\infty} a_r$ must converge ***R1***

Note: If the limit comparison test is used award marks to a maximum of ***R1M1A1M0A0R1***.

[6 marks]

Total [11 marks]

Examiners report

a. [N/A]

b. [N/A]

Consider the differential equation $\frac{dy}{dx} + \frac{x}{x^2+1}y = x$ where $y = 1$ when $x = 0$.

a. Show that $\sqrt{x^2+1}$ is an integrating factor for this differential equation. [4]

b. Solve the differential equation giving your answer in the form $y = f(x)$. [6]

Markscheme

a. **METHOD 1**

integrating factor $= e^{\int \frac{x}{x^2+1}dx}$ ***(M1)***

$\int \frac{x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1)$ ***(M1)***

Note: Award ***M1*** for use of $u = x^2+1$ for example or $\int \frac{f'(x)}{f(x)}dx = \ln f(x)$.

integrating factor $= e^{\frac{1}{2}\ln(x^2+1)}$ ***A1***

$= e^{\ln\left(\sqrt{x^2+1}\right)}$ ***A1***

Note: Award ***A1*** for $e^{\ln\sqrt{u}}$ where $u = x^2+1$.

$= \sqrt{x^2+1}$ ***AG***

METHOD 2

$\frac{d}{dx}\left(y\sqrt{x^2+1}\right) = \frac{dy}{dx}\sqrt{x^2+1} + \frac{x}{\sqrt{x^2+1}}y$ ***M1A1***

$\sqrt{x^2+1}\left(\frac{dy}{dx} + \frac{x}{x^2+1}y\right)$ ***M1A1***

Note: Award ***M1*** for attempting to express in the form $\sqrt{x^2+1} \times (\text{LHS of de})$.

so $\sqrt{x^2+1}$ is an integrating factor for this differential equation ***AG***

[4 marks]

b. $\sqrt{x^2+1}\frac{dy}{dx} + \frac{x}{\sqrt{x^2+1}}y = x\sqrt{x^2+1}$ (or equivalent) ***(M1)***

$\frac{d}{dx}\left(y\sqrt{x^2+1}\right) = x\sqrt{x^2+1}$

$y\sqrt{x^2+1} = \int x\sqrt{x^2+1}dx$ $\left(y = \frac{1}{\sqrt{x^2+1}}\int x\sqrt{x^2+1}dx\right)$ ***A1***

$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$ ***(M1)A1***

Note: Award ***M1*** for using an appropriate substitution.

Note: Condone the absence of C.

substituting $x = 0,\ y = 1 \Rightarrow C = \frac{2}{3}$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$y = \frac{1}{3}(x^2 + 1) + \frac{2}{3\sqrt{x^2+1}}$ $\left(y = \frac{(x^2+1)^{\frac{3}{2}}+2}{3\sqrt{x^2+1}}\right)$ ***A1***

[6 marks]

Examiners report

a. [N/A]
b. [N/A]

---

a. Show that $n! \geqslant 2^{n-1}$, for $n \geqslant 1$. [2]

b. Hence use the comparison test to determine whether the series $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ converges or diverges. [3]

Markscheme

a. for $n \geqslant 1,\ n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1 \geqslant 2 \times 2 \times 2 \ldots 2 \times 2 \times 1 = 2^{n-1}$ ***M1A1***

$\Rightarrow n! \geqslant 2^{n-1}$ for $n \geqslant 1$ ***AG***

[2 marks]

b. $n! \geqslant 2^{n-1} \Rightarrow \frac{1}{n!} \leqslant \frac{1}{2^{n-1}}$ for $n \geqslant 1$ ***A1***

$\sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a positive converging geometric series ***R1***

hence $\sum\limits_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test ***R1***

[3 marks]

Examiners report

a. Part (a) of this question was found challenging by the majority of candidates, a fairly common 'solution' being that the result is true for $n = 1$, 2, 3 and therefore true for all n. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between $\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.

b. Part (a) of this question was found challenging by the majority of candidates, a fairly common 'solution' being that the result is true for $n = 1$, 2, 3 and therefore true for all n. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between $\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.

---

a. Show that the series $\sum\limits_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converges. [3]

b. (i) Show that $\ln(n) + \ln\left(1 + \frac{1}{n}\right) = \ln(n+1)$. [6]

(ii) Using this result, show that an application of the ratio test fails to determine whether or not $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

c. (i) State why the integral test can be used to determine the convergence or divergence of $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$. [8]

(ii) Hence determine the convergence or divergence of $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$.

Markscheme

a. **METHOD 1**

$(0 <)\frac{1}{n^2 \ln(n)} < \frac{1}{n^2}$, (for $n \geq 3$) ***A1***

$\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ converges ***A1***

by the comparison test ($\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ converges implies) $\sum\limits_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$ converges ***R1***

Note: Mention of using the comparison test may have come earlier.

Only award ***R1*** if previous 2 ***A1***s have been awarded.

METHOD 2

$\lim\limits_{n\to\infty}\left(\frac{\frac{1}{n^2 \ln n}}{\frac{1}{n^2}}\right) = \lim\limits_{n\to\infty}\frac{1}{\ln n} = 0$ ***A1***

$\sum\limits_{n=2}^{\infty} \frac{1}{n^2}$ converges ***A1***

by the limit comparison test (if the limit is 0 and the series represented by the denominator converges, then so does the series represented by the numerator, hence) $\sum\limits_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$ converges ***R1***

Note: Mention of using the limit comparison test may come earlier.

Do not award the ***R1*** if incorrect justifications are given, for example the series "converge or diverge together".

Only award ***R1*** if previous 2 ***A1***s have been awarded.

[3 marks]

b. (i) **EITHER**

$\ln(n)+\ln\left(1+\frac{1}{n}\right)=\ln\left(n\left(1+\frac{1}{n}\right)\right)$ ***A1***

OR

$\ln(n)+\ln\left(1+\frac{1}{n}\right)=\ln(n)+\ln\left(\frac{n+1}{n}\right)$

$=\ln(n)+\ln(n+1)-\ln(n)$ ***A1***

THEN

$=\ln(n+1)$ ***AG***

(ii) attempt to use the ratio test $\frac{n}{(n+1)}\frac{\ln(n)}{\ln(n+1)}$ ***M1***

$\frac{n}{n+1}\to 1 \text{ as } n\to\infty$ ***(A1)***

$\frac{\ln(n)}{\ln(n+1)}=\frac{\ln(n)}{\ln(n)+\ln\left(1+\frac{1}{n}\right)}$ ***M1***

$\to 1 \quad (\text{as } n\to\infty)$ ***(A1)***

$\frac{n}{(n+1)}\frac{\ln(n)}{\ln(n+1)}\to 1 \quad (\text{as } n\to\infty)$ hence ratio test is inconclusive ***R1***

Note: A link with the limit equalling 1 and the result being inconclusive needs to be given for ***R1***.

[6 marks]

c. (i) consider $f(x)=\frac{1}{x\ln x} \quad (\text{for } x>1)$ ***A1***

$f(x)$ is continuous and positive ***A1***

and is (monotonically) decreasing ***A1***

Note: If a candidate uses n rather than x, award as follows

$\frac{1}{n\ln n}$ is positive and decreasing ***A1A1***

$\frac{1}{n\ln n}$ is continuous for $n\in\mathbb{R},\ n>1$ ***A1*** (only award this mark if the domain has been explicitly changed).

(ii) consider $\int_2^R \frac{1}{x\ln x}\mathrm{d}x$ ***M1***

$=[\ln(\ln x)]_2^R$ ***(M1)A1***

$\to\infty \text{ as } R\to\infty$ ***R1***

hence series diverges ***A1***

Note: Condone the use of ∞ in place of R.

Note: If the lower limit is not equal to 2, but the expression is integrated correctly award ***M0M1A1R0A0***.

[8 marks]

Total [17 marks]

Examiners report

a. In this part the required test was not given in the question. This led to some students attempting inappropriate methods. When using the comparison or limit comparision test many candidates wrote the incorrect statement $\frac{1}{n^2}$ converges, (p-series) rather than the correct one with $\sum$. This perhaps indicates a lack of understanding of the concepts involved.

b. There were many good, well argued answers to this part. Most candidates recognised the importance of the result in part (i) to find the limit in part (ii). Generally a standard result such as $\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right) = 1$ can simply be quoted, but other limits such as $\lim\limits_{n\to\infty}\left(\frac{\ln n}{\ln(n+1)}\right) = 1$ need to be carefully justified.

c. (i) Candidates need to be aware of the necessary conditions for all the series tests.

(ii) The integration was well done by the candidates. Most also made the correct link between the integral being undefined and the series diverging. In this question it was not necessary to initially take a finite upper limit and the use of ∞ was acceptable. This was due to the command term being 'determine'. In q4b a finite upper limit was required, as the command term was 'show'. To ensure full marks are always awarded candidates should err on the side of caution and always use limit notation when working out indefinite integrals.

The Taylor series of $\sqrt{x}$ about $x = 1$ is given by

$$a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \ldots$$

a. Find the values of $a_0,\ a_1,\ a_2$ and a_3. [6]

b. Hence, or otherwise, find the value of $\lim\limits_{x\to1}\frac{\sqrt{x}-1}{x-1}$. [3]

Markscheme

a. let $f(x) = \sqrt{x},\ f(1) = 1$ ***(A1)***

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}},\ f'(1) = \frac{1}{2}$ ***(A1)***

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}},\ f''(1) = -\frac{1}{4}$ ***(A1)***

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}},\ f'''(1) = \frac{3}{8}$ ***(A1)***

$a_1 = \frac{1}{2}\cdot\frac{1}{1!},\ a_2 = -\frac{1}{4}\cdot\frac{1}{2!},\ a_3 = \frac{3}{8}\cdot\frac{1}{3!}$ ***(M1)***

$a_0 = 1,\ a_1 = \frac{1}{2},\ a_2 = -\frac{1}{8},\ a_3 = \frac{1}{16}$ ***A1***

Note: Accept $y = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 + \ldots$

[6 marks]

b. **METHOD 1**

$\lim\limits_{x\to1}\frac{\sqrt{x}-1}{x-1} = \lim\limits_{x\to1}\frac{\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\ldots}{x-1}$ ***M1***

$= \lim\limits_{x\to1}\left(\frac{1}{2} - \frac{1}{8}(x-1) + \ldots\right)$ ***A1***

$= \frac{1}{2}$ ***A1***

METHOD 2

using l'Hôpital's rule, ***M1***

$\lim\limits_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim\limits_{x \to 1} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1}$ ***A1***

$= \frac{1}{2}$ ***A1***

METHOD 3

$\frac{\sqrt{x}-1}{x+1} = \frac{1}{\sqrt{x}+1}$ ***M1A1***

$\lim\limits_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$ ***A1***

[3 marks]

Examiners report

a. Many candidates achieved full marks on this question but there were still a large minority of candidates who did not seem familiar with the application of Taylor series. Whilst all candidates who responded to this question were aware of the need to use derivatives many did not correctly use factorials to find the required coefficients. It should be noted that the formula for Taylor series appears in the Information Booklet.

b. Many candidates achieved full marks on this question but there were still a large minority of candidates who did not seem familiar with the application of Taylor series. Whilst all candidates who responded to this question were aware of the need to use derivatives many did not correctly use factorials to find the required coefficients. It should be noted that the formula for Taylor series appears in the Information Booklet.

---

a. Use an integrating factor to show that the general solution for $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{x}{t} = -\frac{2}{t}$, $t > 0$ is $x = 2 + ct$, where c is a constant. [4]

The weight in kilograms of a dog, t weeks after being bought from a pet shop, can be modelled by the following function:

$$w(t) = \begin{cases} 2 + ct & 0 \le t \le 5 \\ 16 - \frac{35}{t} & t > 5 \end{cases}.$$

b. Given that $w(t)$ is continuous, find the value of c. [2]

c. Write down [2]

(i) the weight of the dog when bought from the pet shop;

(ii) an upper bound for the weight of the dog.

d. Prove from first principles that $w(t)$ is differentiable at $t = 5$. [6]

Markscheme

a. integrating factor $e^{\int -\frac{1}{t}\mathrm{d}t} = e^{-\ln t}\left(=\frac{1}{t}\right)$ ***M1A1***

$\frac{x}{t} = \int -\frac{2}{t^2}\mathrm{d}t = \frac{2}{t} + c$ ***A1A1***

Note: Award ***A1*** for $\frac{x}{t}$ and ***A1*** for $\frac{2}{t} + c$.

$x = 2 + ct$ ***AG***

[4 marks]

b. given continuity at $x = 5$

$5c + 2 = 16 - \frac{35}{5} \Rightarrow c = \frac{7}{5}$ ***M1A1***

[2 marks]

c. (i) 2 ***A1***

(ii) any value ≥ 16 ***A1***

Note: Accept values less than 16 if fully justified by reference to the maximum age for a dog.

[2 marks]

d. $\lim\limits_{h \to 0-}\left(\frac{\frac{7}{5}(5+h)+2-\frac{7}{5}(5)-2}{h}\right) = \frac{7}{5}$ ***M1A1***

$\lim\limits_{h \to 0+}\left(\frac{16-\frac{35}{5+h}-16+\frac{35}{5}}{h}\right) \quad \left(= \lim\limits_{h \to 0+}\left(\frac{\frac{-35}{5+h}+7}{h}\right)\right)$ ***M1***

$= \lim\limits_{h \to 0+}\left(\frac{\frac{-35+35+7h}{(5+h)}}{h}\right) = \lim\limits_{h \to 0+}\left(\frac{7}{5+h}\right) = \frac{7}{5}$ ***M1A1***

both limits equal so differentiable at $t = 5$ ***R1AG***

Note: The limits $t \to 5$ could also be used.

For each value of $\frac{7}{5}$ obtained by standard differentiation award ***A1***.

To gain the other 4 marks a rigorous explanation must be given on how you can get from the left and right hand derivatives to the derivative.

Note: If the candidate works with t and then substitutes $t = 5$ at the end award as follows

First ***M1*** for using formula with t in the linear case, ***A1*** for $\frac{7}{5}$

Award next 2 method marks even if $t = 5$ not substituted, ***A1*** for $\frac{7}{5}$

[6 marks]

Total [14 marks]

Examiners report

a. This was generally well done. Some candidates did not realize $e^{-\ln t}$ could be simplified to $\frac{1}{t}$.

b. This part was well done by the majority of candidates.

c. This part was well done by the majority of candidates.

d. Some candidates ignored the instruction to prove from first principles and instead used standard differentiation. Some candidates also only found a derivative from one side.

---

a. Consider the power series $\sum_{k=1}^{\infty} k\left(\frac{x}{2}\right)^k$. [10]

(i) Find the radius of convergence.

(ii) Find the interval of convergence.

b. Consider the infinite series $\sum_{k=1}^{\infty} (-1)^{k+1} \times \frac{k}{2k^2+1}$. [5]

(i) Show that the series is convergent.

(ii) Show that the sum to infinity of the series is less than 0.25.

Markscheme

a. (i) consider $\frac{T_{n+1}}{T_n} = \frac{\left|\frac{(n-1)x^{n+1}}{2^{n+1}}\right|}{\left|\frac{nx^n}{2^n}\right|}$ ***M1***

$= \frac{(n+1)|x|}{2n}$ ***A1***

$\to \frac{|x|}{2}$ as $n \to \infty$ ***A1***

the radius of convergence satisfies

$\frac{R}{2} = 1$, *i.e.* $R = 2$ ***A1***

(ii) the series converges for $-2 < x < 2$, we need to consider $x = \pm 2$ ***(R1)***

when $x = 2$, the series is $1 + 2 + 3 + \ldots$ ***A1***

this is divergent for any one of several reasons *e.g.* finding an expression for or a comparison test with the harmonic series or noting that $\lim_{n\to\infty} u_n \neq 0$ etc. ***R1***

when $x = -2$, the series is $-1 + 2 - 3 + 4 \ldots$ ***A1***

this is divergent for any one of several reasons

e.g. partial sums are

$-1,\ 1,\ -2,\ 2,\ -3,\ 3\ldots$ or noting that $\lim_{n\to\infty} u_n \neq 0$ *etc.* ***R1***

the interval of convergence is $-2 < x < 2$ ***A1***

[10 marks]

b. (i) this alternating series is convergent because the moduli of successive terms are monotonic decreasing ***R1***

and the n^{th} term tends to zero as $n \to \infty$ ***R1***

(ii) consider the partial sums

0.333, 0.111, 0.269, 0.148, 0.246 ***M1A1***

since the sum to infinity lies between any pair of successive partial sums, it follows that the sum to infinity lies between 0.148 and 0.246 so that it is less than 0.25 ***R1***

Note: Accept a solution which looks only at 0.333, 0.269, 0.246 and states that these are successive upper bounds.

[5 marks]

Examiners report

a. Most candidates found the radius of convergence correctly but examining the situation when $x = \pm 2$ often ended in loss of marks through inadequate explanations. In (b)(i) many candidates were able to justify the convergence of the given series. In (b)(ii), however, many candidates seemed unaware of the fact the sum to infinity lies between any pair of successive partial sums.

b. Most candidates found the radius of convergence correctly but examining the situation when $x = \pm 2$ often ended in loss of marks through inadequate explanations. In (b)(i) many candidates were able to justify the convergence of the given series. In (b)(ii), however, many candidates seemed unaware of the fact the sum to infinity lies between any pair of successive partial sums.

(a) Show that the solution of the homogeneous differential equation

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + 1,\ x > 0,$

given that $y = 0$ when $x = \mathrm{e}$, is $y = x(\ln x - 1)$.

(b) (i) Determine the first three derivatives of the function $f(x) = x(\ln x - 1)$.

(ii) Hence find the first three non-zero terms of the Taylor series for $f(x)$ about $x = 1$.

Markscheme

(a) **EITHER**

use the substitution $y = vx$

$\frac{\mathrm{d}y}{\mathrm{d}x}x + v = v + 1$ ***M1A1***

$\int \mathrm{d}v = \int \frac{\mathrm{d}x}{x}$

by integration

$v = \frac{y}{x} = \ln x + c$ ***A1***

OR

the equation can be rearranged as first order linear

$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = 1$ ***M1***

the integrating factor I is

$\mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x} = \mathrm{e}^{-\ln x} = \frac{1}{x}$ ***A1***

multiplying by I gives

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}y\right) = \frac{1}{x}$

$\frac{1}{x}y = \ln x + c$ ***A1***

THEN

the condition gives $c = -1$

so the solution is $y = x(\ln x - 1)$ ***AG***

[5 marks]

(b) (i) $f'(x) = \ln x - 1 + 1 = \ln x$ ***A1***

$f''(x) = \frac{1}{x}$ ***A1***

$f'''(x) = -\frac{1}{x^2}$ ***A1***

(ii) the Taylor series about $x = 1$ starts

$f(x) \approx f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} + f'''(1)\frac{(x-1)^3}{3!}$ ***(M1)***

$= -1 + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!}$ ***A1A1A1***

[7 marks]

Total: [12 marks]

Examiners report

Part(a) was well done by many candidates. In part(b)(i), however, it was disappointing to see so many candidates unable to differentiate $x(\ln x - 1)$ correctly. Again, too many candidates were able to quote the general form of a Taylor series expansion, but not how to apply it to the given function.

The function f is defined by $f(x) = (\arcsin x)^2, \ -1 \leqslant x \leqslant 1$.

The function f satisfies the equation $\left(1 - x^2\right) f''(x) - xf'(x) - 2 = 0$.

a. Show that $f'(0) = 0$. [2]

b. By differentiating the above equation twice, show that [4]

$$\left(1 - x^2\right) f^{(4)}(x) - 5xf^{(3)}(x) - 4f''(x) = 0$$

where $f^{(3)}(x)$ and $f^{(4)}(x)$ denote the 3rd and 4th derivative of $f(x)$ respectively.

c. Hence show that the Maclaurin series for $f(x)$ up to and including the term in x^4 is $x^2 + \frac{1}{3}x^4$. [3]

d. Use this series approximation for $f(x)$ with $x = \frac{1}{2}$ to find an approximate value for π^2. [2]

Markscheme

a. $f'(x) = \frac{2\arcsin(x)}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for an attempt at chain rule differentiation.
Award ***M0A0*** for $f'(x) = 2\arcsin(x)$.

$f'(0) = 0$ ***AG***

[2 marks]

b. differentiating gives $(1-x^2)f^{(3)}(x) - 2xf''(x) - f'(x) - xf''(x) (= 0)$ ***M1A1***

differentiating again gives $(1-x^2)f^{(4)}(x) - 2xf^{(3)}(x) - 3f''(x) - 3xf^{(3)}(x) - f''(x) (= 0)$ ***M1A1***

Note: Award ***M1*** for an attempt at product rule differentiation of at least one product in each of the above two lines.
Do not penalise candidates who use poor notation.

$(1-x^2)f^{(4)}(x) - 5xf^{(3)}(x) - 4f''(x) = 0$ ***AG***

[4 marks]

c. attempting to find **one of** $f''(0)$, $f^{(3)}(0)$ or $f^{(4)}(0)$ by substituting $x = 0$ into relevant differential equation(s) ***(M1)***

Note: Condone $f''(0)$ found by calculating $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2\arcsin(x)}{\sqrt{1-x^2}}\right)$ at $x = 0$.

$(f(0) = 0,\ f'(0) = 0)$

$f''(0) = 2$ and $f^{(4)}(0) - 4f''(0) = 0 \Rightarrow f^{(4)}(0) = 8$ ***A1***

$f^{(3)}(0) = 0$ and so $\frac{2}{2!}x^2 + \frac{8}{4!}x^4$ ***A1***

Note: Only award the above ***A1***, for correct first differentiation in part (b) leading to $f^{(3)}(0) = 0$ stated or $f^{(3)}(0) = 0$ seen from use of the general Maclaurin series.
Special Case: Award ***(M1)A0A1*** if $f^{(4)}(0) = 8$ is stated without justification or found by working backwards from the general Maclaurin series.

so the Maclaurin series for $f(x)$ up to and including the term in x^4 is $x^2 + \frac{1}{3}x^4$ ***AG***

[3 marks]

d. substituting $x = \frac{1}{2}$ into $x^2 + \frac{1}{3}x^4$ ***M1***

the series approximation gives a value of $\frac{13}{48}$

so $\pi^2 \simeq \frac{13}{48} \times 36$

$\simeq 9.75 \left(\simeq \frac{39}{4}\right)$ ***A1***

Note: Accept 9.76.

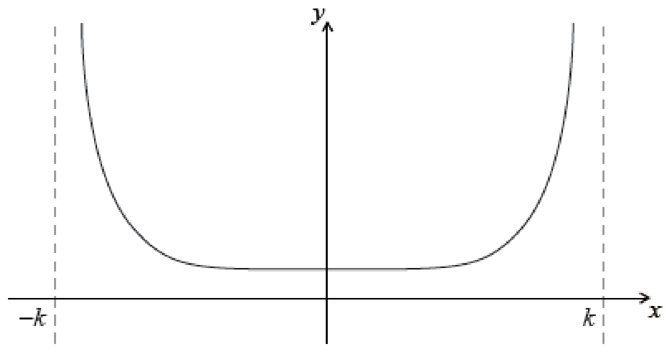
[2 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

A function f is defined in the interval $]-k,\ k[$, where $k > 0$. The gradient function f' exists at each point of the domain of f.

The following diagram shows the graph of $y = f(x)$, its asymptotes and its vertical symmetry axis.



(a) Sketch the graph of $y = f'(x)$.

Let $p(x) = a + bx + cx^2 + dx^3 + \dots$ be the Maclaurin expansion of $f(x)$.

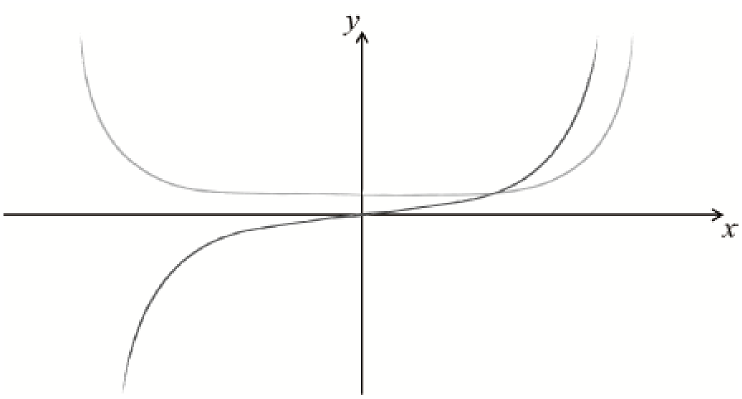
(b) (i) Justify that $a > 0$.

(ii) Write down a condition for the largest set of possible values for each of the parameters b, c and d.

(c) State, with a reason, an upper bound for the radius of convergence.

Markscheme

(a)



A1 for shape, ***A1*** for passing through origin ***A1A1***

Note: Asymptotes not required.

[2 marks]

(b) $p(x) = \underbrace{f(0)}_{a} + \underbrace{f'(0)}_{b}x + \underbrace{\frac{f''(0)}{2!}}_{c}x^2 + \underbrace{\frac{f^{(3)}(0)}{3!}}_{d}x^3 + \dots$

(i) because the y-intercept of f is positive ***R1***

(ii) $b = 0$ ***A1***

$c \geqslant 0$ ***A1A1***

Note: ***A1*** for $>$ and ***A1*** for $=$.

$d = 0$ ***A1***

[5 marks]

(c) as the graph has vertical asymptotes $x = \pm k,\ k > 0,$ ***R1***

the radius of convergence has an upper bound of k ***A1***

Note: Accept $r < k$.

[2 marks]

Examiners report

Overall candidates made good attempts to parts (a) and most candidates realized that the graph contained the origin; however many candidates had difficulty rendering the correct shape of the graph of f'. Part b(i) was also well answered although some candidates where not very clear and digressed a lot. Part (ii) was less successful with most candidates scoring just part of the marks. A small number of candidates answered part (c) correctly with a valid reason.

Consider the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \frac{x^3}{x^2+1}.$$

(a) Find an integrating factor for this differential equation.

(b) Solve the differential equation given that $y = 1$ when $x = 1$, giving your answer in the forms $y = f(x)$.

Markscheme

(a) Rewrite the equation in the form

$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = \frac{x^2}{x^2+1}$ ***M1A1***

Integrating factor $= \mathrm{e}^{\int -\frac{2}{x}\mathrm{d}x}$ ***M1***

$= \mathrm{e}^{-2\ln x}$ ***A1***

$= \frac{1}{x^2}$ ***A1***

Note: Accept $\frac{1}{x^3}$ as applied to the original equation.

[5 marks]

(b) Multiplying the equation,

$\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y = \frac{1}{x^2+1}$ ***(M1)***

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) = \frac{1}{x^2+1}$ ***(M1)(A1)***

$\frac{y}{x^2} = \int \frac{\mathrm{d}x}{x^2+1}$ ***M1***

$= \arctan x + C$ ***A1***

Substitute $x = 1,\ y = 1$. ***M1***

$1 = \frac{\pi}{4} + C \Rightarrow C = 1 - \frac{\pi}{4}$ ***A1***

$y = x^2\left(\arctan x + 1 - \frac{\pi}{4}\right)$ ***A1***

[8 marks]

Total [13 marks]

Examiners report

The response to this question was often disappointing. Many candidates were unable to find the integrating factor successfully.

The real and imaginary parts of a complex number $x + \mathrm{i}y$ are related by the differential equation $(x + y)\frac{\mathrm{d}y}{\mathrm{d}x} + (x - y) = 0$.

By solving the differential equation, given that $y = \sqrt{3}$ when x =1, show that the relationship between the modulus r and the argument θ of the complex number is $r = 2\mathrm{e}^{\frac{\pi}{3} - \theta}$.

Markscheme

$(x + y)\frac{\mathrm{d}y}{\mathrm{d}x} + (x - y) = 0$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - x}{x + y}$

let $y = vx$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***A1***

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx - x}{x + vx}$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v - 1}{v + 1} - v = \frac{v - 1 - v^2 - v}{v + 1} = \frac{-1 - v^2}{1 + v}$ ***A1***

$\int \frac{v + 1}{1 + v^2}\mathrm{d}v = -\int \frac{1}{x}\mathrm{d}x$ ***M1***

$\int \frac{v}{1 + v^2}\mathrm{d}v + \int \frac{1}{1 + v^2}\mathrm{d}v = -\int \frac{1}{x}\mathrm{d}x$ ***M1***

$\Rightarrow \frac{1}{2}\ln\left|1 + v^2\right| + \arctan v = -\ln|x| + k$ ***A1A1***

Notes: Award ***A1*** for $\frac{1}{2}\ln\left|1 + v^2\right|$, ***A1*** for the other two terms.

Do not penalize missing k or missing modulus signs at this stage.

$\Rightarrow \frac{1}{2}\ln\left|1 + \frac{y^2}{x^2}\right| + \arctan\frac{y}{x} = -\ln|x| + k$ ***M1***

$\Rightarrow \frac{1}{2}\ln 4 + \arctan\sqrt{3} = -\ln 1 + k$ ***(M1)***

$\Rightarrow k = \ln 2 + \frac{\pi}{3}$ ***A1***

$\Rightarrow \frac{1}{2}\ln\left|1 + \frac{y^2}{x^2}\right| + \arctan\frac{y}{x} = -\ln|x| + \ln 2 + \frac{\pi}{3}$

attempt to combine logarithms ***M1***

$\Rightarrow \frac{1}{2}\ln\left|\frac{y^2 + x^2}{x^2}\right| + \frac{1}{2}\ln\left|x^2\right| = \ln 2 + \frac{\pi}{3} - \arctan\frac{y}{x}$

$\Rightarrow \frac{1}{2}\ln\left|y^2 + x^2\right| = \ln 2 + \frac{\pi}{3} - \arctan\frac{y}{x}$ ***(A1)***

$\Rightarrow \sqrt{y^2 + x^2} = \mathrm{e}^{\ln 2 + \frac{\pi}{3}\arctan\frac{y}{x}}$ ***(A1)***

$\Rightarrow \sqrt{y^2 + x^2} = \mathrm{e}^{\ln 2} \times \mathrm{e}^{\frac{\pi}{3} - \arctan\frac{y}{x}}$ ***A1***

$\Rightarrow r = 2\mathrm{e}^{\frac{\pi}{3} - \theta}$ ***AG***

[15 marks]

Examiners report

Most candidates realised that this was a homogeneous differential equation and that the substitution $y = vx$ was the way forward. Many of these candidates reached as far as separating the variables correctly but integrating $\frac{v+1}{v^2+1}$ proved to be too difficult for many candidates – most failed to realise that the expression had to be split into two separate integrals. Some candidates successfully evaluated the arbitrary constant but the combination of logs and the subsequent algebra necessary to obtain the final result proved to be beyond the majority of candidates.

---

a. Consider the differential equation [3]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right),\ x > 0.$$

Use the substitution $y = vx$ to show that the general solution of this differential equation is

$$\int \frac{\mathrm{d}v}{f(v) - v} = \ln x + \text{ Constant}.$$

b. Hence, or otherwise, solve the differential equation [10]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + 3xy + y^2}{x^2},\ x > 0,$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = g(x)$.

Markscheme

a. $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1***

the differential equation becomes

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = f(v)$ ***A1***

$\int \frac{\mathrm{d}v}{f(v)-v} = \int \frac{\mathrm{d}v}{x}$ ***A1***

integrating, Constant $\int \frac{\mathrm{d}v}{f(v)-v} = \ln x + \text{ Constant}$ ***AG***

[3 marks]

b. **EITHER**

$f(v) = 1 + 3v + v^2$ ***(A1)***

$\left(\int \frac{\mathrm{d}v}{f(v)-v} =\right) \int \frac{\mathrm{d}v}{1+3v+v^2-v} = \ln x + C$ ***M1A1***

$\int \frac{\mathrm{d}v}{(1+v)^2} = (\ln x + C)$ ***A1***

Note: ***A1*** is for correct factorization.

$-\frac{1}{1+v}\ (= \ln x + C)$ ***A1***

OR

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 3v + v^2$ ***A1***

$\int \frac{\mathrm{d}v}{1+2v+v^2} = \int \frac{1}{x}\mathrm{d}x$ ***M1***

$\int \frac{\mathrm{d}v}{(1+v)^2} \left(= \int \frac{1}{x}\mathrm{d}x\right)$ ***(A1)***

Note: ***A1*** is for correct factorization.

$-\frac{1}{1+v} = \ln x(+C)$ ***A1A1***

THEN

substitute $y = 1$ or $v = 1$ when $x = 1$ ***(M1)***

therefore $C = -\frac{1}{2}$ ***A1***

Note: This ***A1*** can be awarded anywhere in their solution.

substituting for v,

$-\frac{1}{\left(1+\frac{y}{x}\right)} = \ln x - \frac{1}{2}$ ***M1***

Note: Award for correct substitution of $\frac{y}{x}$ into their expression.

$1 + \frac{y}{x} = \frac{1}{\frac{1}{2}-\ln x}$ ***(A1)***

Note: Award for any rearrangement of a correct expression that has y in the numerator.

$y = x\left(\frac{1}{\left(\frac{1}{2}-\ln x\right)} - 1\right)$ (or equivalent) ***A1***

$\left(= x\left(\frac{1+2\ln x}{1-2\ln x}\right)\right)$

[10 marks]

Examiners report

a. [N/A]
b. [N/A]

Solve the differential equation

$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 3xy + 2x^2$

given that $y = -1$ when $x = 1$. Give your answer in the form $y = f(x)$.

Markscheme

put $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1*** substituting, ***M1***

$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{v^2x^2+3vx^2+2x^2}{x^2}\ (=v^2+3v+2)$ ***(A1)***

$x\frac{\mathrm{d}v}{\mathrm{d}x}=v^2+2v+2$ ***A1***

$\int\frac{\mathrm{d}v}{v^2+2v+2}=\int\frac{\mathrm{d}x}{x}$ ***M1***

$\int\frac{\mathrm{d}v}{(v+1)^2+1}=\int\frac{\mathrm{d}x}{x}$ ***(A1)***

$\arctan(v+1)=\ln x+c$ ***A1***

Note: Condone absence of c at **this** stage.

$\arctan(\frac{y}{x}+1)=\ln x+c$ ***M1***

When $x=1, y=-1$ ***M1***

$c=0$ ***A1***

$\frac{y}{x}+1=\tan\ln x$

$y=x(\tan\ln x-1)$ ***A1***

[11 marks]

Examiners report

Most candidates recognised this differential equation as one in which the substitution $y=vx$ would be helpful and many reached the stage of separating the variables. However, the integration of $\frac{1}{v^2+2v+2}$ proved beyond many candidates who failed to realise that completing the square would lead to an arctan integral. This highlights the importance of students having a full understanding of the core calculus if they are studying this option.

Let $f(x)=\mathrm{e}^x\sin x$.

a. Show that $f''(x)=2\left(f'(x)-f(x)\right)$. [4]

b. By further differentiation of the result in part (a) , find the Maclaurin expansion of $f(x)$, as far as the term in x^5. [6]

Markscheme

a. $f'(x)=\mathrm{e}^x\sin x+\mathrm{e}^x\cos x$ ***M1A1***

$f''(x)=\mathrm{e}^x\sin x+\mathrm{e}^x\cos x-\mathrm{e}^x\sin x+\mathrm{e}^x\cos x=2\mathrm{e}^x\cos x$ ***A1***

$=2\left(\mathrm{e}^x\sin x+\mathrm{e}^x\cos x-\mathrm{e}^x\sin x\right)$ ***M1***

$=2\left(f'(x)-f(x)\right)$ ***AG***

[4 marks]

b. $f(0) = 0,\ f'(0) = 1,\ f''(0) = 2(1 - 0) = 2$ ***(M1)A1***

Note: Award ***M1*** for attempt to find $f(0)$, $f'(0)$ and $f''(0)$.

$f'''(x) = 2\left(f''(x) - f'(x)\right)$ ***(M1)***

$f'''(0) = 2(2 - 1) = 2,\ f^{IV}(0) = 2(2 - 2) = 0,\ f^{V}(0) = 2(0 - 2) = -4$ ***A1***

so $f(x) = x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 - \frac{4}{5!} + \ldots$ ***(M1)A1***

$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots$

[6 marks]

Total [10 marks]

Examiners report

a. [N/A]
b. [N/A]

Find the exact value of $\int_0^\infty \frac{\mathrm{d}x}{(x+2)(2x+1)}$.

Markscheme

Let $\frac{1}{(x+2)(2x+1)} = \frac{A}{x+2} + \frac{B}{2x+1} = \frac{A(2x+1)+B(x+2)}{(x+2)(2x+1)}$ ***M1A1***

$x = -2 \to A = -\frac{1}{3}$ ***A1***

$x = -\frac{1}{2} \to B = \frac{2}{3}$ ***A1*** ***N3***

$I = \frac{1}{3}\int_0^h \left[\frac{2}{(2x+1)} - \frac{1}{(x+2)}\right]\mathrm{d}x$ ***M1***

$= \frac{1}{3}[\ln(2x+1) - \ln(x+2)]_0^h$ ***A1***

$= \frac{1}{3}\left[\lim_{h\to\infty}\left(\ln\left(\frac{2h+1}{h+2}\right)\right) - \ln\frac{1}{2}\right]$ ***A1***

$= \frac{1}{3}\left(\ln 2 - \ln\frac{1}{2}\right)$ ***A1***

$= \frac{2}{3}\ln 2$ ***A1***

Note: If the logarithms are not combined in the third from last line the last three ***A1*** marks cannot be awarded.

Total [9 marks]

Examiners report

Not a difficult question but combination of the logarithms obtained by integration was often replaced by a spurious argument with infinities to get an answer. $\log(\infty + 1)$ was often seen.

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \cos^2 x$, given that $y = 2$ when $x = 0$.

a. Use Euler's method with a step length of 0.1 to find an approximation to the value of y when $x = 0.3$. [5]

b. (i) Show that the integrating factor for solving the differential equation is $\sec x$. [10]

(ii) Hence solve the differential equation, giving your answer in the form $y = f(x)$.

Markscheme

a. use of $y \to y + \frac{h\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

x	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{h\mathrm{d}y}{\mathrm{d}x}$
0	2	1	0.1
0.1	2.1	0.7793304775	0.07793304775
0.2	2.17793304775	0.5190416116	0.05190416116
0.3	2.229837209		

Note: Award ***A1*** for $y(0.1)$ and ***A1*** for $y(0.2)$

$y(0.3) = 2.23$ ***A2***

[5 marks]

b. (i) $\mathrm{IF} = \mathrm{e}^{\left(\int \tan x \mathrm{d}x\right)}$ ***(M1)***

$\mathrm{IF} = \mathrm{e}^{\left(\int \frac{\sin x}{\cos x}\mathrm{d}x\right)}$ ***(M1)***

Note: Only one of the two ***(M1)*** above may be implied.

$= \mathrm{e}^{(-\ln\cos x)}$ (or $\mathrm{e}^{(\ln\sec x)}$) ***A1***

$= \sec x$ ***AG***

(ii) multiplying by the IF ***(M1)***

$\sec x\frac{\mathrm{d}y}{\mathrm{d}x} + y\sec x\tan x = \cos x$ ***(A1)***

$\frac{\mathrm{d}}{\mathrm{d}x}(y\sec x) = \cos x$ ***(A1)***

$y\sec x = \sin x + c$ ***A1A1***

putting $x = 0,\ y = 2 \Rightarrow c = 2$

$y = \cos x(\sin x + 2)$ ***A1***

[10 marks]

Examiners report

a. Most candidates knew Euler's method and were able to apply it to the differential equation to answer part (a). Some candidates who knew Euler's method completed one iteration too many to arrive at an incorrect answer. Surprisingly few candidates were able to efficiently use their GDCs to answer this question and this led to many final answers that were incorrect due to rounding errors.

b. Most candidates were able to correctly derive the Integration Factor in part (b) but some lost marks due to not showing all the steps that would be expected in a "show that" question. The differential equation was solved correctly by a significant number of candidates but there were errors when candidates multiplied by $\sec x$ before the inclusion of the arbitrary constant.

Let $g(x) = \sin x^2$, where $x \in \mathbb{R}$.

a. Using the result $\lim\limits_{t \to 0} \frac{\sin t}{t} = 1$, or otherwise, calculate $\lim\limits_{x \to 0} \frac{g(2x) - g(3x)}{4x^2}$. [4]

b. Use the Maclaurin series of $\sin x$ to show that $g(x) = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$ [2]

c. Hence determine the minimum number of terms of the expansion of $g(x)$ required to approximate the value of $\int_0^1 g(x)\mathrm{d}x$ to four decimal places. [7]

Markscheme

a. **METHOD 1**

$\lim\limits_{x \to 0} \frac{\sin 4x^2 - \sin 9x^2}{4x^2}$ ***M1***

$= \lim\limits_{x \to 0} \frac{\sin 4x^2}{4x^2} - \frac{9}{4} \lim\limits_{x \to 0} \frac{\sin 9x^2}{9x^2}$ ***A1A1***

$= 1 - \frac{9}{4} \times 1 = -\frac{5}{4}$ **A1**

METHOD 2

$\lim\limits_{x \to 0} \frac{\sin 4x^2 - \sin 9x^2}{4x^2}$ ***M1***

$= \lim\limits_{x \to 0} \frac{8x \cos 4x^2 - 18x \cos 9x^2}{8x}$ ***M1A1***

$= \frac{8 - 18}{8} = -\frac{10}{8} = -\frac{5}{4}$ ***A1***

[4 marks]

b. since $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$ $\left(\text{or } \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$ ***(M1)***

$\sin x^2 = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{(2n+1)!}$ $\left(\text{or } \sin x = \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots\right)$ ***A1***

$g(x) = \sin x^2 = \sum\limits_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$ ***AG***

[2 marks]

c. let $I = \int_0^1 \sin x^2 \mathrm{d}x$

$= \sum\limits_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_0^1 x^{4n+2} \mathrm{d}x$ $\left(\int_0^1 \frac{x^2}{1!} \mathrm{d}x - \int_0^1 \frac{x^6}{3!} \mathrm{d}x + \int_0^1 \frac{x^{10}}{5!} \mathrm{d}x - \ldots\right)$ ***M1***

$= \sum\limits_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{[x^{4n+3}]_0^1}{(4n+3)}$ $\left(\left[\frac{x^3}{3 \times 1!}\right]_0^1 - \left[\frac{x^7}{7 \times 3!}\right]_0^1 + \left[\frac{x^{11}}{11 \times 5!}\right]_0^1 - \ldots\right)$ ***M1***

$= \sum\limits_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!(4n+3)}$ $\left(\frac{1}{3 \times 1!} - \frac{1}{7 \times 3!} + \frac{1}{11 \times 5!} - \ldots\right)$ ***A1***

$= \sum\limits_{n=0}^{\infty} (-1)^n a_n$ where $a_n = \frac{1}{(4n+3)(2n+1)!} > 0$ for all $n \in \mathbb{N}$

as $\{a_n\}$ is decreasing the sum of the alternating series $\sum\limits_{n=0}^{\infty} (-1)^n a_n$

lies between $\sum\limits_{n=0}^{N} (-1)^n a_n$ and $\sum\limits_{n=0}^{N} (-1)^n a_n \pm a_{N+1}$ ***R1***

hence for four decimal place accuracy, we need $|a_{N+1}| < 0.00005$ ***M1***

N	$\lvert a_{N+1} \rvert$
1	$\frac{1}{11(5!)} = 0.0000757576$
2	$\frac{1}{15(7!)} = 0.0000132275$

since $a_{2+1} < 0.00005$ ***R1***

so $N = 2$ (or 3 terms) ***A1***

[7 marks]

Examiners report

a. Part (a) of this question was accessible to the vast majority of candidates, who recognised that L'Hôpital's rule could be used. Most candidates were successful in finding the limit, with some making calculation errors. Candidates that attempted to use $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ or a combination of this result and L'Hôpital's rule were less successful.

b. In part (b) most candidates showed to be familiar with the substitution given and were successful in showing the result.

c. Very few candidates were able to do part (c) successfully. Most used trial and error to arrive at the answer.

(a) Using the Maclaurin series for the function e^x, write down the first four terms of the Maclaurin series for $\mathrm{e}^{-\frac{x^2}{2}}$.

(b) Hence find the first four terms of the series for $\int_0^x \mathrm{e}^{-\frac{u^2}{2}} \mathrm{d}u$.

(c) Use the result from part (b) to find an approximate value for $\frac{1}{\sqrt{2\pi}} \int_0^1 \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$.

Markscheme

(a) $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

putting $x = \frac{-x^2}{2}$ ***(M1)***

$\mathrm{e}^{-\frac{x^2}{2}} \approx 1 - \frac{x^2}{2} + \frac{x^4}{2^2 \times 2!} - \frac{x^6}{2^3 \times 3!} \approx \left(1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}\right)$ ***A2***

[3 marks]

(b) $\int_0^x \mathrm{e}^{-\frac{u^2}{2}} \mathrm{d}u \approx \left[u - \frac{u^3}{3 \times 2} + \frac{u^5}{5 \times 2^2 \times 2!} - \frac{u^7}{7 \times 2^3 \times 3!}\right]_0^x$ ***M1(A1)***

$= x - \frac{x^3}{3 \times 2} + \frac{x^5}{5 \times 2^2 \times 2!} - \frac{x^7}{7 \times 2^3 \times 3!}$ ***A1***

$\left(= x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336}\right)$

[3 marks]

(c) putting $x = 1$ in part (b) gives $\int_0^1 \mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x \approx 0.85535\ldots$ ***(M1)(A1)***

$\frac{1}{\sqrt{2\pi}}\int_0^1 \mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x \approx 0.341$ ***A1***

[3 marks]

Total [9 marks]

Examiners report

This was one of the most successfully answered questions. Some candidates however failed to use the data booklet for the expansion of the series, thereby wasting valuable time.

Consider the infinite series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}x^n$.

a. Find the radius of convergence. [4]

b. Find the interval of convergence. [3]

c. Given that $x = -0.1$, find the sum of the series correct to three significant figures. [4]

Markscheme

a. $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2 x^{n+1}}{2^{n+1}}}{\frac{n^2 x^n}{2^n}}$ ***M1***

$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \times \frac{x}{2}$ ***A1***

$= \frac{x}{2}$ (since $\lim \to \frac{x}{2}$ as $n \to \infty$) ***A1***

the radius of convergence R is found by equating this limit to 1, giving $R = 2$ ***A1***

[4 marks]

b. when $x = 2$, the series is $\sum n^2$ which is divergent because the terms do not converge to 0 ***R1***

when $x = -2$, the series is $\sum (-1)^n n^2$ which is divergent because the terms do not converge to 0 ***R1***

the interval of convergence is $]-2,\ 2[$ ***A1***

[3 marks]

c. putting $x = -0.1$, ***(M1)***

for any correct partial sum ***(A1)***

– 0.05

– 0.04

– 0.041125

– 0.041025

– 0.0410328 ***(A1)***

the sum is – 0.0410 correct to 3 significant figures ***A1***

[4 marks]

Examiners report

a. It was pleasing that most candidates were aware of the Radius of Convergence and Interval of Convergence required by parts (a) and (b) of this problem. Many candidates correctly handled the use of the Ratio Test for convergence and there was also the use of Cauchy's n^{th} root test by a small number of candidates to solve part (a). Candidates need to take care to justify correctly the divergence or convergence of series when finding the Interval of Convergence.

b. It was pleasing that most candidates were aware of the Radius of Convergence and Interval of Convergence required by parts (a) and (b) of this problem. Many candidates correctly handled the use of the Ratio Test for convergence and there was also the use of Cauchy's n^{th} root test by a small number of candidates to solve part (a). Candidates need to take care to justify correctly the divergence or convergence of series when finding the Interval of Convergence.

c. The summation of the series in part (c) was poorly handled by a significant number of candidates, which was surprising on what was expected to be quite a straightforward problem. Again efficient use of the GDC seemed to be a problem. A number of candidates found the correct sum but not to the required accuracy.

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x+\sqrt{xy}}$, for $x,\ y > 0$.

(a) Use Euler's method starting at the point $(x,\ y) = (1,\ 2)$, with interval $h = 0.2$, to find an approximate value of y when $x = 1.6$.

(b) Use the substitution $y = vx$ to show that $x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{1+\sqrt{v}} - v$.

(c) (i) Hence find the solution of the differential equation in the form $f(x,\ y) = 0$, given that $y = 2$ when $x = 1$.

(ii) Find the value of y when $x = 1.6$.

Markscheme

(a) let $f(x,\ y) = \frac{y}{x+\sqrt{xy}}$

$y(1.2) = y(1) + 0.2f(1,\ 2)\ (= 2 + 0.1656\ldots)$ ***(M2)(A1)***

$= 2.1656\ldots$ ***A1***

$y(1.4) = 2.1656\ldots + 0.2f(1.2,\ 2.1256\ldots)\ (= 2.1656\ldots + 0.1540\ldots)$ ***(M1)***

Note: ***M1*** is for attempt to apply formula using point $(1.2,\ y(1.2))$.

$= 2.3197\ldots$ ***A1***

$y(1.6) = 2.3197\ldots + 0.2f(1.4,\ 2.3197\ldots)\ (= 2.3297\ldots + 0.1448\ldots)$

$= 2.46\ \ (3\text{sf})$ ***A1*** ***N3***

[7 marks]

(b) $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x+\sqrt{xy}} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx}{x+\sqrt{vx^2}}$ ***M1***

$\Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{vx}{x+x\sqrt{v}}$ (as $x > 0$) ***A1***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{1+\sqrt{v}} - v$ ***AG***

[3 marks]

(c) (i) $x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v}{1+\sqrt{v}} - v$

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-v\sqrt{v}}{1+\sqrt{v}} \Rightarrow \frac{1+\sqrt{v}}{-v\sqrt{v}}\mathrm{d}v = \frac{1}{x}\mathrm{d}x$ ***M1***

$\int \frac{1+\sqrt{v}}{-v\sqrt{v}}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ ***(M1)***

$\frac{2}{\sqrt{v}} - \ln v = \ln x + C$ ***A1A1***

Note: Do not penalize absence of $+C$ at this stage; ignore use of absolute values on \(v\)and \(x\) (which are positive anyway).

$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} = \ln x + C$ as $y = vx \Rightarrow v = \frac{y}{x}$ ***M1***

$y = 2$ when $x = 1 \Rightarrow \sqrt{2} - \ln 2 = 0 + C$ ***M1***

$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} = \ln x + \sqrt{2} - \ln 2$

$2\sqrt{\frac{x}{y}} - \ln\frac{y}{x} - \ln x - \sqrt{2} + \ln 2 = 0 \quad \left(2\sqrt{\frac{x}{y}} - \ln y - \sqrt{2} + \ln 2 = 0\right)$ ***A1***

(ii) $2\sqrt{\frac{1.6}{y}} - \ln\frac{y}{1.6} - \ln 1.6 - \sqrt{2} + \ln 2 = 0$ ***(M1)***

$y = 2.45$ ***A1***

[9 marks]

Examiners report

Part (a) was well answered by most candidates. In a few cases calculation errors and early rounding errors prevented candidates from achieving full marks, but most candidates scored at least a few marks here. In part (b) some candidates failed to convincingly show the given result. Part (c) proved to be a hard question for many candidates and a significant number of candidates had difficulty manipulating the algebraic expression, and either had the incorrect expression to integrate, or incorrectly integrated the correct expression. Many candidates reached as far as separating the variables correctly but integrating proved to be too difficult for many of them although most realised that the expression on v had to be split into two separate integrals. Most candidates made good attempts to evaluate the arbitrary constant and arrived at a correct or almost correct expression (sign errors were a common error) which allowed follow through for part b (ii). In some cases however the expression obtained was too simple or was omitted and it was not possible to grant follow through marks.

---

a. Show that $y = \frac{1}{x}\int f(x)\mathrm{d}x$ is a solution of the differential equation [3]

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x),\ x > 0.$$

b. Hence solve $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^{-\frac{1}{2}},\ x > 0$, given that $y = 2$ when $x = 4$. [5]

Markscheme

a. **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}\int f(x)\mathrm{d}x + \frac{1}{x}f(x)$ ***M1M1A1***

$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x),\ x > 0$ ***AG***

Note: ***M1*** for use of product rule, ***M1*** for use of the fundamental theorem of calculus, ***A1*** for all correct.

METHOD 2

$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x)$

$\frac{\mathrm{d}(xy)}{\mathrm{d}x} = f(x)$ ***(M1)***

$xy = \int f(x)\mathrm{d}x$ ***M1A1***

$y = \frac{1}{x}\int f(x)\mathrm{d}x$ ***AG***

[3 marks]

b. $y = \frac{1}{x}\left(2x^{\frac{1}{2}} + c\right)$ ***A1A1***

Note: ***A1*** for correct expression apart from the constant, ***A1*** for including the constant in the correct position.

attempt to use the boundary condition ***M1***

$c = 4$ ***A1***

$y = \frac{1}{x}\left(2x^{\frac{1}{2}} + 4\right)$ ***A1***

Note: Condone use of integrating factor.

[5 marks]

Total [8 marks]

Examiners report

a. This question allowed for several different approaches. The most common of these was the use of the integrating factor (even though that just took you in a circle). Other candidates substituted the solution into the differential equation and others multiplied the solution by x and then used the product rule to obtain the differential equation. All these were acceptable.

b. This was a straightforward question. Some candidates failed to use the hint of 'hence', and worked from the beginning using the integrating factor. A surprising number made basic algebra errors such as putting the $+c$ term in the wrong place and so not dividing it by χ.

Consider the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{x^2 - y^2},\ x > 0,\ x^2 > y^2.$$

a. Show that this is a homogeneous differential equation. [1]

b. Find the general solution, giving your answer in the form $y = f(x)$. [7]

Markscheme

a. the equation can be rewritten as

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+\sqrt{x^2-y^2}}{x} = \frac{y}{x} + \sqrt{1-\left(\frac{y}{x}\right)^2}$ ***A1***

so the differential equation is homogeneous ***AG***

[1 mark]

b. put $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***M1A1***

substituting,

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + \sqrt{1-v^2}$ ***M1***

$\int \frac{\mathrm{d}v}{\sqrt{1-v^2}} = \int \frac{\mathrm{d}x}{x}$ ***M1***

$\arcsin v = \ln x + C$ ***A1***

$\frac{y}{x} = \sin(\ln x + C)$ ***A1***

$y = x\sin(\ln x + C)$ ***A1***

[7 marks]

Examiners report

a. [N/A]

b. [N/A]

The function $f(x) = \frac{1+ax}{1+bx}$ can be expanded as a power series in x, within its radius of convergence R, in the form $f(x) \equiv 1 + \sum\limits_{n=1}^{\infty} c_n x^n$.

(a) (i) Show that $c_n = (-b)^{n-1}(a-b)$.

(ii) State the value of R.

(b) Determine the values of a and b for which the expansion of $f(x)$ agrees with that of e^x up to and including the term in x^2 .

(c) Hence find a rational approximation to $e^{\frac{1}{3}}$.

Markscheme

(a) (i) $f(x) = (1+ax)(1+bx)^{-1}$

$= (1+ax)(1-bx+\ldots(-1)^n b^n x^n + \ldots$ ***M1A1***

it follows that

$c_n = (-1)^n b^n + (-1)^{n-1} a b^{n-1}$ ***M1A1***

$= (-b)^{n-1}(a-b)$ ***AG***

(ii) $\mathrm{R}=\frac{1}{|b|}$ ***A1***

[5 marks]

(b) to agree up to quadratic terms requires

$1=-b+a,\ \frac{1}{2}=b^2-ab$ ***M1A1A1***

from which $a=-b=\frac{1}{2}$ ***A1***

[4 marks]

(c) $\mathrm{e}^x\approx\frac{1+0.5x}{1-0.5x}$ ***A1***

putting $x=\frac{1}{3}$ ***M1***

$\mathrm{e}^{\frac{1}{3}}\approx\frac{\left(1+\frac{1}{6}\right)}{\left(1-\frac{1}{6}\right)}=\frac{7}{5}$ ***A1***

[3 marks]

Total [12 marks]

Examiners report

Most candidates failed to realize that the first step was to write *f*(*x*) as $(1+ax)(1+bx)^{-1}$. Given the displayed answer to part(a), many candidates successfully tackled part(b). Few understood the meaning of the ‘hence’ in part(c).

---

a. Find the first three terms of the Maclaurin series for $\ln(1+\mathrm{e}^x)$. [6]

b. Hence, or otherwise, determine the value of $\lim\limits_{x\to0}\frac{2\ln(1+\mathrm{e}^x)-x-\ln 4}{x^2}$. [4]

Markscheme

a. **METHOD 1**

$f(x)=\ln(1+\mathrm{e}^x);\ f(0)=\ln 2$ ***A1***

$f'(x)=\frac{\mathrm{e}^x}{1+\mathrm{e}^x};\ f'(0)=\frac{1}{2}$ ***A1***

Note: Award ***A0*** for $f'(x)=\frac{1}{1+\mathrm{e}^x};\ f'(0)=\frac{1}{2}$

$f''(x)=\frac{\mathrm{e}^x(1+\mathrm{e}^x)-\mathrm{e}^{2x}}{(1+\mathrm{e}^x)^2};\ f''(0)=\frac{1}{4}$ ***M1A1***

Note: Award ***M0A0*** for $f''(x)$ if $f'(x)=\frac{1}{1+\mathrm{e}^x}$ is used

$\ln(1+\mathrm{e}^x)=\ln 2+\frac{1}{2}x+\frac{1}{8}x^2+\ldots$ ***M1A1***

[6 marks]

METHOD 2

$\ln(1+e^x) = \ln(1+1+x+\frac{1}{2}x^2+\ldots)$ **M1A1**

$= \ln 2 + \ln(1+\frac{1}{2}x+\frac{1}{4}x^2+\ldots)$ **A1**

$= \ln 2 + \left(\frac{1}{2}x+\frac{1}{4}x^2+\ldots\right) - \frac{1}{2}\left(\frac{1}{2}x+\frac{1}{4}x^2+\ldots\right)^2 + \ldots$ ***A1***

$= \ln 2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 + \ldots$ ***A1***

$= \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \ldots$ ***A1***

[6 marks]

b. **METHOD 1**

$\lim\limits_{x\to 0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2} = \lim\limits_{x\to 0}\frac{2\ln 2+x+\frac{x^2}{4}+x^3\text{ terms \& above}-x-\ln 4}{x^2}$ ***M1A1***

$= \lim\limits_{x\to 0}\left(\frac{1}{4}+\text{powers of }x\right) = \frac{1}{4}$ ***M1A1***

Note: Accept + … as evidence of recognition of cubic and higher powers.

Note: Award ***M1AOM1A0*** for a solution which omits the cubic and higher powers.

[4 marks]

METHOD 2

using l'Hôpital's Rule

$\lim\limits_{x\to 0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2} = \lim\limits_{x\to 0}\frac{2e^x\div(1+e^x)-1}{2x}$ ***M1A1***

$= \lim\limits_{x\to 0}\frac{2e^x\div(1+e^x)^2}{2} = \frac{1}{4}$ ***M1A1***

[4 marks]

Examiners report

a. In (a), candidates who found the series by successive differentiation were generally successful, the most common error being to state that the derivative of $\ln(1+e^x)$ is $(1+e^x)^{-1}$. Some candidates assumed the series for $\ln(1+x)$ and e^x attempted to combine them. This was accepted as an alternative solution but candidates using this method were often unable to obtain the required series.

b. In (b), candidates were equally split between using the series or using l'Hopital's rule to find the limit. Both methods were fairly successful, but a number of candidates forgot that if a series was used, there had to be a recognition that it was not a finite series.

The function *f* is defined by $f(x) = e^{(e^x-1)}$.

(a) Assuming the Maclaurin series for e^x , show that the Maclaurin series for $f(x)$

is $1 + x + x^2 + \frac{5}{6}x^3 + \ldots$.

(b) Hence or otherwise find the value of $\lim\limits_{x\to 0}\frac{f(x)-1}{f'(x)-1}$.

Markscheme

(a) $e^x - 1 = x + \frac{x^2}{2} + \frac{x^2}{6} + \ldots$ ***A1***

$e^{e^x-1} = 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{\left(x+\frac{x^2}{2}+\frac{x^3}{6}\right)^2}{2} + \frac{\left(x+\frac{x^2}{2}+\frac{x^3}{6}\right)^3}{6} + \ldots$ ***M1A1***

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^3}{6} + \ldots$ ***M1A1***

$= 1 + x + x^2 + \frac{5}{6}x^3 + \ldots$ ***AG***

[5 marks]

(b) **EITHER**

$f'(x) = 1 + 2x + \frac{5x^2}{2} + \ldots$ ***A1***

$\frac{f(x)-1}{f'(x)-1} = \frac{x+x^2+5x^3/6+\ldots}{2x+5x^2/2+\ldots}$ ***M1A1***

$= \frac{1+x+\ldots}{2+5x/2+\ldots}$ ***A1***

$\to \frac{1}{2}$ as $x \to 0$ ***A1***

[5 marks]

OR

using l'Hopital's rule, ***M1***

$\lim_{x\to 0} \frac{e^{(e^x-1)}-1}{e^{(e^x-1)}-1'-1} = \lim_{x\to 0} \frac{e^{(e^x-1)}-1}{e^{(e^x+x-1)}-1}$ ***M1A1***

$= \lim_{x\to 0} \frac{e^{(e^x+x-1)}}{e^{(e^x+x-1)}\times(e^x+1)}$ ***A1***

$= \frac{1}{2}$ ***A1***

[5 marks]

Total [10 marks]

Examiners report

Many candidates obtained the required series by finding the values of successive derivatives at $x = 0$, failing to realise that the intention was to start with the exponential series and replace x by the series for $e^x - 1$. Candidates who did this were given partial credit for using this method. Part (b) was reasonably well answered using a variety of methods.

---

a. Determine whether the series $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ is convergent or divergent. [5]

b. Show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent. [7]

Markscheme

a. comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ ***A1***

using the limit comparison test ***(M1)***

$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}\left(=\lim_{x\to 0}\frac{\sin x}{x}\right)=1$ ***M1A1***

since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $\sum_{n=1}^{\infty}\sin\frac{1}{n}$ diverges ***A1***

[5 marks]

b. using integral test ***(M1)***

let $u=\ln x$ ***(M1)***

$\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$\int\frac{1}{x(\ln x)^2}\mathrm{d}x=\int\frac{1}{u^2}\mathrm{d}u=-\frac{1}{u}\left(=-\frac{1}{\ln x}\right)$ ***A1***

$\Rightarrow\int_2^{\infty}\frac{1}{x(\ln x)^2}\mathrm{d}x=\lim_{a\to\infty}\left[-\frac{1}{\ln x}\right]_2^a$

$=\lim_{a\to\infty}\left(-\frac{1}{\ln a}+\frac{1}{\ln 2}\right)$ ***(M1)(A1)***

as $a\to\infty,\ -\frac{1}{\ln a}\to 0$ ***A1***

$\Rightarrow\int_2^{\infty}\frac{1}{x(\ln x)^2}\mathrm{d}x=\frac{1}{\ln 2}$

hence the series is convergent ***AG***

[7 marks]

Examiners report

a. This question was found to be the hardest on the paper, with only the best candidates gaining full marks on it. Part (a) was very poorly done with a significant number of candidates unable to start the question. More students recognised part (b) as an integral test, but often could not progress beyond this. In many cases, students appeared to be guessing at what might constitute a valid test.

b. This question was found to be the hardest on the paper, with only the best candidates gaining full marks on it. Part (a) was very poorly done with a significant number of candidates unable to start the question. More students recognised part (b) as an integral test, but often could not progress beyond this. In many cases, students appeared to be guessing at what might constitute a valid test.

Find the general solution of the differential equation $t\frac{\mathrm{d}y}{\mathrm{d}t}=\cos t-2y$, for $t>0$.

Markscheme

recognise equation as first order linear and attempt to find the IF ***M1***

$\mathrm{IF}=\mathrm{e}^{\int\frac{2}{t}\mathrm{d}t}=t^2$ ***A1***
solution $yt^2=\int t\cos t\mathrm{d}t$ ***M1A1***
using integration by parts with the correct choice of u and v ***(M1)***
$\int t\cos t\mathrm{d}t=t\sin t+\cos t(+C)$ ***A1***
obtain $y=\frac{\sin t}{t}+\frac{\cos t+C}{t^2}$ ***A1***
[7 marks]

Examiners report

Perhaps a small number of candidates were put off by the unusual choice of variables but in most instances it seemed that candidates who recognised the need for an integration factor could make a good attempt at this problem. Candidates who were not able to simplify the integrating factor from $e^{2\ln t}$ to t^2 rarely gained full marks. A significant number of candidates did not gain the final mark due to a lack of an arbitrary constant or not dividing the constant by the integration factor.

Consider the infinite series

$$\frac{1}{2\ln 2}-\frac{1}{3\ln 3}+\frac{1}{4\ln 4}-\frac{1}{5\ln 5}+\dots.$$

(a) Show that the series converges.

(b) Determine if the series converges absolutely or conditionally.

Markscheme

(a) applying the alternating series test as $\forall n\geqslant 2,\ \frac{1}{n\ln n}\in\mathbb{R}^+$ ***M1***

$\forall n,\ \frac{1}{(n+1)\ln(n+1)}\leqslant\frac{1}{n\ln n}$ ***A1***

$\lim\limits_{n\to\infty}\frac{1}{n\ln n}=0$ ***A1***

hence, by the alternating series test, the series converges ***R1***

[4 marks]

(b) as $\frac{1}{x\ln x}$ is a continuous decreasing function, apply the integral test to determine if it converges absolutely ***(M1)***

$\int_2^\infty\frac{1}{x\ln x}\mathrm{d}x=\lim\limits_{b\to\infty}\int_2^b\frac{1}{x\ln x}\mathrm{d}x$ ***M1A1***

let $u=\ln x$ then $\mathrm{d}u=\frac{1}{x}\mathrm{d}x$ ***(M1)A1***

$\int\frac{1}{u}\mathrm{d}u=\ln u$ ***(A1)***

hence, $\lim\limits_{b\to\infty}\int_2^b\frac{1}{x\ln x}\mathrm{d}x=\lim\limits_{b\to\infty}\left[\ln(\ln x)\right]_2^b$ which does not exist ***M1A1A1***

hence, the series does not converge absolutely ***(A1)***

the series converges conditionally ***A1***

[11 marks]

Total [15 marks]

Examiners report

Part (a) was answered well by many candidates who attempted this question. In part (b), those who applied the integral test were mainly successful, but too many failed to supply the justification for its use, and proper conclusions.

---

a. Find the value of $\int\limits_4^\infty \frac{1}{x^3}\mathrm{d}x$. [3]

b. Illustrate graphically the inequality $\sum\limits_{n=5}^\infty \frac{1}{n^3} < \int\limits_4^\infty \frac{1}{x^3}\mathrm{d}x < \sum\limits_{n=4}^\infty \frac{1}{n^3}$. [4]

c. Hence write down a lower bound for $\sum\limits_{n=4}^\infty \frac{1}{n^3}$. [1]

d. Find an upper bound for $\sum\limits_{n=4}^\infty \frac{1}{n^3}$. [3]

Markscheme

a. $\int\limits_4^\infty \frac{1}{x^3}\mathrm{d}x = \lim\limits_{R\to\infty}\int\limits_4^R \frac{1}{x^3}\mathrm{d}x$ ***(A1)***

Note: The above ***A1*** for using a limit can be awarded at any stage.
Condone the use of $\lim\limits_{x\to\infty}$.

Do not award this mark to candidates who use ∞ as the upper limit throughout.

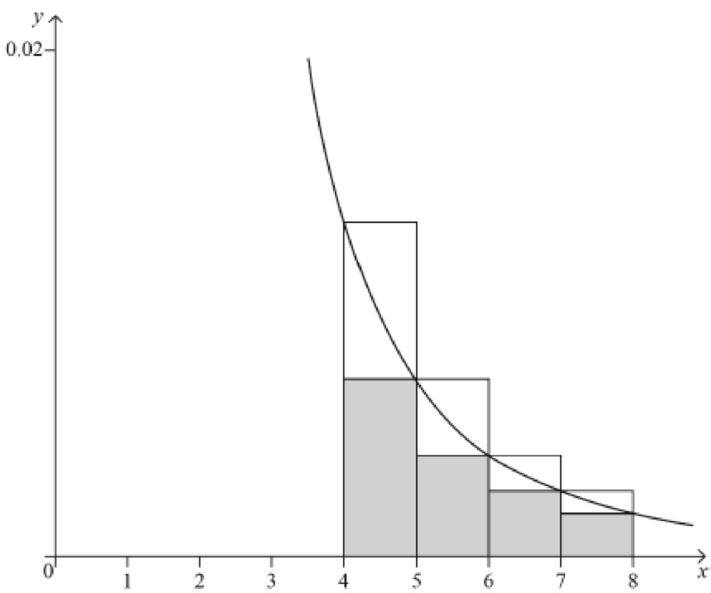
$= \lim\limits_{R\to\infty}\left[-\frac{1}{2}x^{-2}\right]_4^R \left(= \left[-\frac{1}{2}x^{-2}\right]_4^\infty\right)$ ***M1***

$= \lim\limits_{R\to\infty}\left(-\frac{1}{2}\left(R^{-2}-4^{-2}\right)\right)$

$= \frac{1}{32}$ ***A1***

[3 marks]

b.



A1A1A1A1

A1 for the curve
A1 for rectangles starting at $x = 4$
A1 for at least three upper rectangles
A1 for at least three lower rectangles

Note: Award ***A0A1*** for two upper rectangles and two lower rectangles.

sum of areas of the lower rectangles < the area under the curve < the sum of the areas of the upper rectangles so

$\sum\limits_{n=5}^\infty \frac{1}{n^3} < \int\limits_4^\infty \frac{1}{x^3}\mathrm{d}x < \sum\limits_{n=4}^\infty \frac{1}{n^3}$ ***AG***

[4 marks]

c. a lower bound is $\frac{1}{32}$ ***A1***

Note: Allow **FT** from part (a).

[1 mark]

d. **METHOD 1**

$\sum_{n=5}^{\infty} \frac{1}{n^3} < \frac{1}{32}$ ***(M1)***

$\frac{1}{64} + \sum_{n=5}^{\infty} \frac{1}{n^3} = \frac{1}{32} + \frac{1}{64}$ ***(M1)***

$\sum_{n=4}^{\infty} \frac{1}{n^3} < \frac{3}{64}$, an upper bound ***A1***

Note: Allow ***FT*** from part (a).

METHOD 2

changing the lower limit in the inequality in part (b) gives

$\sum_{n=4}^{\infty} \frac{1}{n^3} < \int_3^{\infty} \frac{1}{x^3} \mathrm{d}x \left(< \sum_{n=3}^{\infty} \frac{1}{n^3} \right)$ ***(A1)***

$\sum_{n=4}^{\infty} \frac{1}{n^3} < \lim_{R \to \infty} \left[-\frac{1}{2} x^{-2} \right]_3^R$ ***(M1)***

$\sum_{n=4}^{\infty} \frac{1}{n^3} < \frac{1}{18}$, an upper bound ***A1***

Note: Condone candidates who do not use a limit.

[3 marks]

Examiners report

a. [N/A]
b. [N/A]
c. [N/A]
d. [N/A]

Consider the infinite series $\sum_{n=1}^{\infty} \frac{2}{n^2+3n}$.

Use a comparison test to show that the series converges.

Markscheme

EITHER

$\sum_{n=1}^{\infty} \frac{2}{n^2+3n} < \sum_{n=1}^{\infty} \frac{2}{n^2}$ ***M1***

which is convergent ***A1***

the given series is therefore convergent using the comparison test ***AG***

OR

$\lim_{n \to \infty} \frac{\frac{2}{n^2+3n}}{\frac{1}{n^2}} = 2$ ***M1A1***

the given series is therefore convergent using the limit comparison test ***AG***

[2 marks]

Examiners report

Most candidates were able to answer part (a) and many gained a fully correct answer. A number of candidates ignored the factor 2 in the numerator and this led to candidates being penalised. In some cases candidates were not able to identify an appropriate series to compare with. Most candidates used the Comparison test rather than the Limit comparison test.

The general term of a sequence $\{a_n\}$ is given by the formula $a_n = \frac{e^n + 2^n}{2e^n}$, $n \in \mathbb{Z}^+$.

(a) Determine whether the sequence $\{a_n\}$ is decreasing or increasing.

(b) Show that the sequence $\{a_n\}$ is convergent and find the limit L.

(c) Find the smallest value of $N \in \mathbb{Z}^+$ such that $|a_n - L| < 0.001$, for all $n \geqslant N$.

Markscheme

(a) $a_n = \frac{e^n+2^n}{2e^n} = \frac{1}{2} + \frac{1}{2}\left(\frac{2}{e}\right)^2 > \frac{1}{2} + \frac{1}{2}\left(\frac{2}{e}\right)^{n+1} = a_{n+1}$ ***M1A1***

the sequence is decreasing (as terms are positive) ***A1***

Note: Accept reference to the sum of a constant and a decreasing geometric sequence.

Note: Accept use of derivative of $f(x) = \frac{e^x+2x}{2e^x}$ (and condone use of n) and graphical methods (graph of the sequence or graph of corresponding function f or graph of its derivative f').

Accept a list of consecutive terms of the sequence clearly decreasing (*eg* $0.8678\ldots,\ 0.77067\ldots,\ \ldots$).

[3 marks]

(b) $L = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2} + \frac{1}{2}\left(\frac{2}{e}\right)^n = \frac{1}{2} + \frac{1}{2} \times 0 = \frac{1}{2}$ ***M1A1***

[2 marks]

(c) $\left|a_n - \frac{1}{2}\right| = \left|\frac{1}{2} + \frac{1}{2}\left(\frac{2}{e}\right)^n - \frac{1}{2}\right| = \left|\frac{1}{2}\left(\frac{2}{e}\right)^n\right| < \frac{1}{1000}$ ***M1***

EITHER

$\Rightarrow \left(\frac{e}{2}\right)^n > 500$ ***(A1)***

$\Rightarrow n > 20.25\ldots$ ***(A1)***

OR

$\Rightarrow \left(\frac{2}{e}\right)^n < 500$

$\Rightarrow n > 20.25\ldots$ ***(A1)(A1)***

Note: *A1* for correct inequality; *A1* for correct value.

THEN

therefore $N = 21$ ***A1***

[4 marks]

Examiners report

Most candidates were successful in answering part (a) using a variety of methods. The majority of candidates scored some marks, if not full marks. Surprisingly, some candidates did not have the correct graph for the function the sequence represents. They obviously did not enter it correctly into their GDCs. Others used one of the two definitions for showing that a sequence is increasing/decreasing, but made mistakes with the algebraic manipulation of the expression, thereby arriving at an incorrect answer. Part (b) was less well answered with many candidates ignoring the command terms 'show that' and 'find' and just writing down the value of the limit. Some candidates attempted to use convergence tests for series with this sequence. Part (c) of this question was found challenging by the majority of candidates due to difficulties in solving inequalities involving absolute value.

Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \times 2^n}$.

a. Find the radius of convergence of the series. [7]

b. Hence deduce the interval of convergence. [4]

Markscheme

a. using the ratio test (and absolute convergence implies convergence) ***(M1)***

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n x^n}{(n)2^n}}\right| \quad \textit{A1A1}$$

Note: Award ***A1*** for numerator, ***A1*** for denominator.

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\times x^{n+1}\times n\times 2^n}{(-1)^n\times(n+1)\times 2^{n+1}\times x^n}\right|$$

$$= \lim_{n\to\infty}\frac{n}{2(n+1)}|x| \quad \textit{(A1)}$$

$$= \frac{|x|}{2} \quad \textit{A1}$$

for convergence we require $\frac{|x|}{2} < 1$ ***M1***

$\Rightarrow |x| < 2$

hence radius of convergence is 2 ***A1***

[7 marks]

b. we now need to consider what happens when $x = \pm 2$ ***(M1)***

when $x = 2$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent (by the alternating series test) ***A1***

when $x = -2$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent ***A1***

hence interval of convergence is $]-2,\ 2]$ ***A1***

[4 marks]

Examiners report

a. Most candidates were able to start (a) and a majority gained a fully correct answer. A number of candidates were careless with using the absolute value sign and with dealing with the negative signs and in the more extreme cases this led to candidates being penalised. Part (b) caused more difficulties, with many candidates appearing to know what to do, but then not succeeding in doing it or in not understanding the significance of the answer gained.

b. Most candidates were able to start (a) and a majority gained a fully correct answer. A number of candidates were careless with using the absolute value sign and with dealing with the negative signs and in the more extreme cases this led to candidates being penalised. Part (b) caused more difficulties, with many candidates appearing to know what to do, but then not succeeding in doing it or in not understanding the significance of the answer gained.

Solve the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + xy + 4x^2,$$

given that $y = 2$ when $x = 1$. Give your answer in the form $y = f(x)$.

Markscheme

put $y = vx$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(M1)***

the equation becomes $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 + v + 4$ ***A1***

$\int \frac{\mathrm{d}v}{v^2+4} = \int \frac{\mathrm{d}x}{x}$ ***A1***

$\frac{1}{2}\arctan\left(\frac{v}{2}\right) = \ln x + C$ ***A1A1***

substituting $(x,\ v) = (1,\ 2)$

$C = \frac{\pi}{8}$ ***M1A1***

the solution is

$\arctan\left(\frac{y}{2x}\right) = 2\ln x + \frac{\pi}{4}$ ***A1***

$y = 2x\tan\left(2\ln x + \frac{\pi}{4}\right)$ ***A1***

[9 marks]

Examiners report

Most candidates recognised this differential equation as one in which the substitution $y = vx$ would be helpful and many carried the method through to a successful conclusion. The most common error seen was an incorrect integration of $\frac{1}{4+v^2}$ with partial fractions and/or a logarithmic evaluation seen. Some candidates failed to include an arbitrary constant which led to a loss of marks later on.

The function f is defined by $f(x) = e^{-x}\cos x + x - 1$.

By finding a suitable number of derivatives of f, determine the first non-zero term in its Maclaurin series.

Markscheme

$f(0) = 0$ ***A1***

$f'(x) = -e^{-x}\cos x - e^{-x}\sin x + 1$ ***M1A1***

$f'(0) = 0$ ***(M1)***

$f''(x) = 2e^{-x}\sin x$ ***A1***

$f''(0) = 0$

$f^{(3)}(x) = -2e^{-x}\sin x + 2e^{-x}\cos x$ ***A1***

$f^{(3)}(0) = 2$

the first non-zero term is $\frac{2x^3}{3!}$ $\left(= \frac{x^3}{3}\right)$ ***A1***

Note: Award no marks for using known series.

[7 marks]

Examiners report

Most students had a good understanding of the techniques involved with this question. A surprising number forgot to show $f(0) = 0$. Some candidates did not simplify the second derivative which created extra work and increased the chance of errors being made.

Consider the differential equation $\frac{dy}{dx} = \frac{y^2 + x^2}{2x^2}$ for which $y = -1$ when $x = 1$.

(a) Use Euler's method with a step length of 0.25 to find an estimate for the value of y when $x = 2$.

(b) (i) Solve the differential equation giving your answer in the form $y = f(x)$.

(ii) Find the value of y when $x = 2$.

Markscheme

(a) Using an increment of 0.25 in the x-values ***A1***

n	x_n	y_n	$f(x_n, y_n)$	$hf(x_n, y_n)$	$y_{n+1} = y_n + hf(x_n, y_n)$	
0	1	−1	1	0.25	−0.75	***(M1)A1***
1	1.25	−0.75	0.68	0.17	−0.58	***A1***
2	1.5	−0.58	0.574756	0.143689	−0.4363...	***A1***
3	1.75	−0.436311	0.531080	0.132770	−0.3035...	***A1***

Note: The ***A1*** marks are awarded for final column.

$\Rightarrow y(2) \approx -0.304$ ***A1***

[7 marks]

(b) (i) let $y = vx$ ***M1***

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$\Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2x^2+x^2}{2x^2}$ ***(M1)***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1-2v+v^2}{2}$ ***(A1)***

$\Rightarrow x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{(1-v)^2}{2}$ ***A1***

$\Rightarrow \int \frac{2}{(1-v)^2}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ ***M1***

$\Rightarrow 2(1-v)^{-1} = \ln x + c$ ***A1A1***

$\Rightarrow \frac{2}{1-\frac{y}{x}} = \ln x + c$

when $x = 1,\ y = -1 \Rightarrow c = 1$ ***M1A1***

$\Rightarrow \frac{2x}{x-y} = \ln x + 1$

$\Rightarrow y = x - \frac{2x}{1+\ln x}\ \left(= \frac{x\ln x - x}{1+\ln x}\right)$ ***M1A1***

(ii) when $x = 2,\ y = -0.362$ $\left(\text{accept } 2 - \frac{4}{1+\ln 2}\right)$ ***A1***

[13 marks]

Total [20 marks]

Examiners report

Part (a) was well done by many candidates, but a number were penalised for not using a sufficient number of significant figures. Part (b) was started by the majority of candidates, but only the better candidates were able to reach the end. Many were unable to complete the question correctly because they did not know what to do with the substitution $y = vx$ and because of arithmetic errors and algebraic errors.

---

a. Given that $y = \ln\left(\frac{1+\mathrm{e}^{-x}}{2}\right)$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^{-y}}{2} - 1$. [5]

b. Hence, by repeated differentiation of the above differential equation, find the Maclaurin series for y as far as the term in x^3, showing that two of the terms are zero. [11]

Markscheme

a. **METHOD 1**

$y = \ln\left(\frac{1+\mathrm{e}^{-x}}{2}\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\mathrm{e}^{-x}}{2(1+\mathrm{e}^{-x})} = \frac{-\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}$ ***M1A1*** now $\frac{1+e^{-x}}{2} = e^y$ ***M1***

$\Rightarrow 1 + e^{-x} = 2e^y$

$\Rightarrow e^{-x} = 2e^y - 1$ ***(A1)***

$\Rightarrow \frac{dy}{dx} = \frac{-2e^y+1}{2e^y}$ ***(A1)***

Note: Only one of the two above ***A1*** marks may be implied.

$\Rightarrow \frac{dy}{dx} = \frac{e^{-y}}{2} = -1$ ***AG***

Note: Candidates may find $\frac{dy}{dx}$ as a function of x and then work backwards from the given answer. Award full marks if done correctly.

METHOD 2

$y = \ln\left(\frac{1+e^{-x}}{2}\right)$

$\Rightarrow e^y = \frac{1+e^{-x}}{2}$ ***M1***

$\Rightarrow e^{-x} = 2e^y - 1$

$\Rightarrow x = -\ln(2e^y - 1)$ ***A1***

$\Rightarrow \frac{dx}{dy} = -\frac{1}{2e^y-1} \times 2e^y$ ***M1A1***

$\Rightarrow \frac{dy}{dx} = \frac{2e^y-1}{-2e^y}$ ***A1***

$\Rightarrow \frac{dy}{dx} = \frac{e^{-y}}{2} - 1$ ***AG***

[5 marks]

b. **METHOD 1**

when $x = 0,\ y = \ln 1 = 0$ ***A1***

when $x = 0,\ \frac{dy}{dx} = \frac{1}{2} - 1 = -\frac{1}{2}$ ***A1***

$\frac{d^2y}{dx^2} = -\frac{e^{-y}}{2}\frac{dy}{dx}$ ***M1A1***

when $x = 0,\ \frac{d^2y}{dx^2} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ ***A1***

$\frac{d^3y}{dx^3} = \frac{e^{-y}}{2}\left(\frac{dy}{dx}\right)^2 - \frac{e^{-y}}{2}\frac{d^2y}{dx^2}$ ***M1A1A1***

when $x = 0,\ \frac{d^3y}{dx^3} = \frac{1}{2} \times \frac{1}{4} - \frac{1}{2} \times \frac{1}{4} = 0$ ***A1***

$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$\Rightarrow y = 0 - \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots$ ***(M1)A1***

two of the above terms are zero ***AG***

METHOD 2

when $x = 0,\ y = \ln 1 = 0$ ***A1***

when $x = 0,\ \frac{dy}{dx} = \frac{1}{2} - 1 = -\frac{1}{2}$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-\mathrm{e}^{-y}}{2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\mathrm{e}^{-y}}{2}\left(\frac{\mathrm{e}^{-y}}{2} - 1\right) = \frac{-\mathrm{e}^{2y}}{4} + \frac{\mathrm{e}^{-y}}{2}$ ***M1A1***

when $x = 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$ ***A1***

$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \left(\frac{\mathrm{e}^{-2y}}{2} - \frac{\mathrm{e}^{-y}}{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1A1***

when $x = 0$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\frac{1}{2} \times \left(\frac{1}{2} - \frac{1}{2}\right) = 0$ ***A1***

$y = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$\Rightarrow y = 0 - \frac{1}{2}x + \frac{1}{8}x^2 + 0x^3 + \ldots$ ***(M1)A1***

two of the above terms are zero ***AG***

[11 marks]

Examiners report

a. Many candidates were successful in (a) with a variety of methods seen. In (b) the use of the chain rule was often omitted when differentiating e^{-y} with respect to x. A number of candidates tried to repeatedly differentiate the original expression, which was not what was asked for, although partial credit was given for this. In this case, they often found problems in simplifying the algebra.

b. Many candidates were successful in (a) with a variety of methods seen. In (b) the use of the chain rule was often omitted when differentiating e^{-y} with respect to x. A number of candidates tried to repeatedly differentiate the original expression, which was not what was asked for, although partial credit was given for this. In this case, they often found problems in simplifying the algebra.

Each term of the power series $\frac{1}{1\times2} + \frac{1}{4\times5}x + \frac{1}{7\times8}x^2 + \frac{1}{10\times11}x^3 + \ldots$ has the form $\frac{1}{b(n)\times c(n)}x^n$, where $b(n)$ and $c(n)$ are linear functions of n.

(a) Find the functions $b(n)$ and $c(n)$.

(b) Find the radius of convergence.

(c) Find the interval of convergence.

Markscheme

(a) $b(n) = 3n + 1$ ***A1***

$c(n) = 3n + 2$ ***A1***

Note: $b(n)$ and $c(n)$ may be reversed.

[2 marks]

(b) consider the ratio of the $(n+1)^{\text{th}}$ and n^{th} terms: ***M1***

$\frac{3n+1}{3n+4} \times \frac{3n+2}{3n+5} \times \frac{x^{n+1}}{x^n}$ ***A1***

$\lim_{n\to0} \frac{3n+1}{3n+4} \times \frac{3n+2}{3n+5} \times \frac{x^{n+1}}{x^n}x$ ***A1***

radius of convergence: $R = 1$ ***A1***

[4 marks]

(c) any attempt to study the series for $x = -1$ or $x = 1$ ***(M1)***

converges for $x = 1$ by comparing with p-series $\sum \frac{1}{n^2}$ ***R1***
attempt to use the alternating series test for $x = -1$ ***(M1)***

Note: At least one of the conditions below needs to be attempted for ***M1*****.**

$|\text{terms}| \approx \frac{1}{9n^2} \to 0$ and terms decrease monotonically in absolute value ***A1***
series converges for $x = -1$ ***R1***
interval of convergence: $[-1,\ 1]$ ***A1***

Note: Award the ***R1***s only if an attempt to corresponding correct test is made;
award the final ***A1*** only if at least one of the ***R1***s is awarded;
Accept study of absolute convergence at end points.

[6 marks]

Examiners report

[N/A]

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} - 2y^2 = \mathrm{e}^x$ and $y = 1$ when $x = 0$, use Euler's method with a step length of 0.1 to find an approximation for the value of y when $x = 0.4$. Give all intermediate values with maximum possible accuracy.

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x + 2y^2$ ***(A1)***

x	y	$\mathrm{d}y/\mathrm{d}x$	δy	
0	1	3	0.3	***M1A1***
0.1	1.3	4.485170918	0.4485170918	***A1***
0.2	1.7485170918	7.336026799	0.7336026799	***A1***
0.3	2.482119772	13.67169593	1.367169593	***A1***
0.4	3.849289365			***A1***

required approximation = 3.85 ***A1***

[8 marks]

Examiners report

Most candidates seemed familiar with Euler's method. The most common way of losing marks was either to round intermediate answers to insufficient accuracy despite the advice in the question or simply to make an arithmetic error. Many candidates were given an accuracy penalty for not rounding their answer to three significant figures.

---

a. Consider the functions $f(x) = (\ln x)^2,\ x > 1$ and $g(x) = \ln(f(x)),\ x > 1$. [5]

(i) Find $f'(x)$.

(ii) Find $g'(x)$.

(iii) Hence, show that $g(x)$ is increasing on $]1,\ \infty[$.

b. Consider the differential equation [12]

$$(\ln x)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{2x-1}{(\ln x)},\ x > 1.$$

(i) Find the general solution of the differential equation in the form $y = h(x)$.

(ii) Show that the particular solution passing through the point with coordinates $\left(\mathrm{e},\ \mathrm{e}^2\right)$ is given by $y = \frac{x^2 - x + \mathrm{e}}{(\ln x)^2}$.

(iii) Sketch the graph of your solution for $x > 1$, clearly indicating any asymptotes and any maximum or minimum points.

Markscheme

a. (i) attempt at chain rule ***(M1)***

$f'(x) = \frac{2\ln x}{x}$ ***A1***

(ii) attempt at chain rule ***(M1)***

$g'(x) = \frac{2}{x\ln x}$ ***A1***

(iii) $g'(x)$ is positive on $]1,\ \infty[$ ***A1***

so $g(x)$ is increasing on $]1,\ \infty[$ ***AG***

[5 marks]

b. (i) rearrange in standard form:

$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x\ln x}y = \frac{2x-1}{(\ln x)^2},\ x > 1$ ***(A1)***

integrating factor:

$\mathrm{e}^{\int \frac{2}{x\ln x}\mathrm{d}x}$ ***(M1)***

$= \mathrm{e}^{\ln\left((\ln x)^2\right)}$

$= (\ln x)^2$ ***(A1)***

multiply by integrating factor ***(M1)***

$(\ln x)^2\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2\ln x}{x}y = 2x - 1$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(y(\ln x)^2\right) = 2x - 1$ $\left(\text{or } y(\ln x)^2 = \int 2x - 1\mathrm{d}x\right)$ ***M1***

attempt to integrate: ***M1***

$(\ln x)^2 y = x^2 - x + c$

$y = \frac{x^2 - x + c}{(\ln x)^2}$ ***A1***

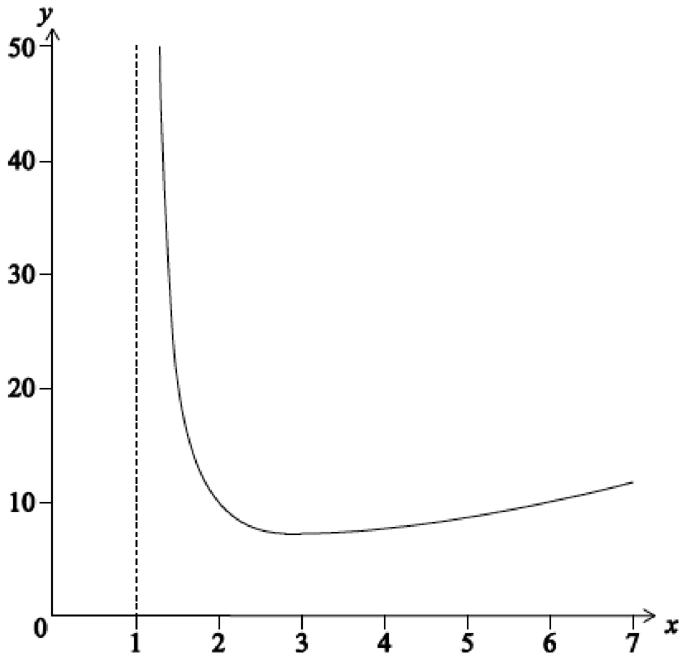
(ii) attempt to use the point $\left(\mathrm{e},\ \mathrm{e}^2\right)$ to determine c: ***M1***

eg, $(\ln \mathrm{e})^2\mathrm{e}^2 = \mathrm{e}^2 - \mathrm{e} + \mathrm{c}$ or $\mathrm{e}^2 = \frac{\mathrm{e}^2 - \mathrm{e} + \mathrm{c}}{(\ln \mathrm{e})^2}$ or $\mathrm{e}^2 = \mathrm{e}^2 - \mathrm{e} + \mathrm{c}$

$\mathrm{c} = \mathrm{e}$ ***A1***

$y = \frac{x^2 - x + \mathrm{e}}{(\ln x)^2}$ ***AG***

(iii)



graph with correct shape ***A1***

minimum at $x = 3.1$ (accept answers to a minimum of 2 s.f) ***A1***

asymptote shown at $x = 1$ ***A1***

Note: *y*-coordinate of minimum not required for ***A1***;

Equation of asymptote not required for ***A1*** if VA appears on the sketch.

Award ***A0*** for asymptotes if more than one asymptote are shown

[12 marks]

Examiners report

a. [N/A]

b. [N/A]

---

a. Using the integral test, show that $\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$ is convergent. [4]

b. (i) Show, by means of a diagram, that $\sum_{n=1}^{\infty} \frac{1}{4n^2+1} < \frac{1}{4\times1^2+1} + \int_1^{\infty} \frac{1}{4x^2+1} \mathrm{d}x$. [4]

(ii) Hence find an upper bound for $\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$

Markscheme

a. $\int \frac{1}{4x^2+1} \mathrm{d}x = \frac{1}{2} \arctan 2x + k$ ***(M1)(A1)***

Note: Do not penalize the absence of "+*k*".

$\int_1^{\infty} \frac{1}{4x^2+1} \mathrm{d}x = \frac{1}{2} \lim_{a \to \infty} [\arctan 2x]_1^a$ ***(M1)***

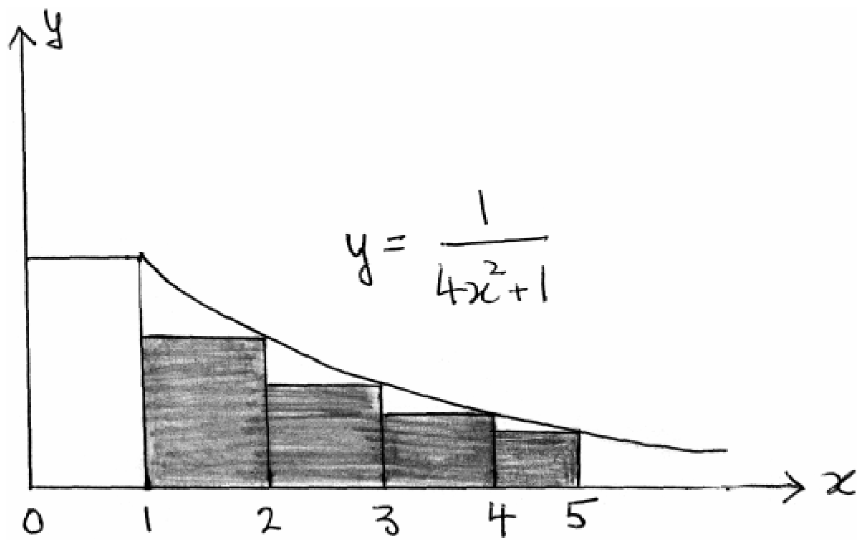
Note: Accept $\frac{1}{2}[\arctan 2x]_1^\infty$.

$= \frac{1}{2}\left(\frac{\pi}{2} - \arctan 2\right) \quad (= 0.232)$ ***A1***

hence the series converges ***AG***

[4 marks]

b. (i)



A2

The shaded rectangles lie within the area below the graph so that $\sum\limits_{n=2}^{\infty} \frac{1}{4n^2+1} < \int_1^\infty \frac{1}{4x^2+1}\mathrm{d}x$. Adding the first term in the series, $\frac{1}{4\times 1^2+1}$, gives $\sum\limits_{n=1}^{\infty} \frac{1}{4n^2+1} < \frac{1}{4\times 1^2+1} + \int_1^\infty \frac{1}{4x^2+1}\mathrm{d}x$ ***R1AG***

(ii) upper bound $= \frac{1}{5} + \frac{1}{2}\left(\frac{\pi}{2} - \arctan 2\right) \quad (= 0.432)$ ***A1***

[4 marks]

Examiners report

a. This proved to be a hard question for most candidates. A number of fully correct answers to (a) were seen, but a significant number were unable to integrate $\frac{1}{4x^2+1}$ successfully. Part (b) was found the hardest by candidates with most candidates unable to draw a relevant diagram, without which the proof of the inequality was virtually impossible.

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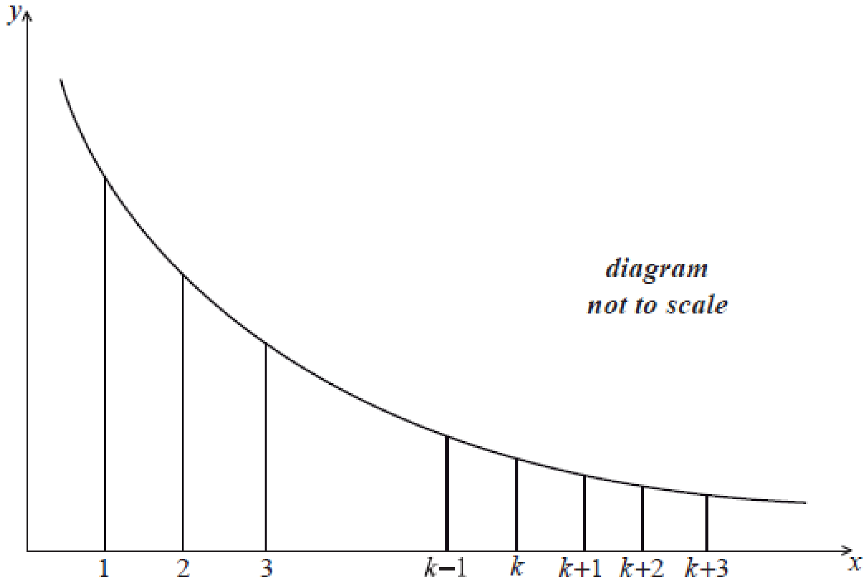
---

a. Prove that $\lim\limits_{H\to\infty} \int_a^H \frac{1}{x^2}\mathrm{d}x$ exists and find its value in terms of a (where $a \in \mathbb{R}^+$). [3]

b. Use the integral test to prove that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges. [3]

c. Let $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = L$. [6]

The diagram below shows the graph of $y = \frac{1}{x^2}$.



(i) Shade suitable regions on a copy of the diagram above and show that $\sum\limits_{n=1}^{k} \frac{1}{n^2} + \int_{k+1}^{\infty} \frac{1}{x^2}\mathrm{d}x < L$.

(ii) Similarly shade suitable regions on another copy of the diagram above and show that $L < \sum\limits_{n=1}^{k} \frac{1}{n^2} + \int_{k}^{\infty} \frac{1}{x^2}\mathrm{d}x$.

d. Hence show that $\sum\limits_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k+1} < L < \sum\limits_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k}$ [2]

e. You are given that $L = \frac{\pi^2}{6}$. [3]

By taking $k = 4$, use the upper bound and lower bound for L to find an upper bound and lower bound for π . Give your bounds to three significant figures.

Markscheme

a. $\lim\limits_{H \to \infty} \int_a^H \frac{1}{x^2}\mathrm{d}x = \lim\limits_{H \to \infty} \left[\frac{-1}{x}\right]_a^H$ ***A1***

$\lim\limits_{H \to \infty} \left(\frac{-1}{H} + \frac{1}{a}\right)$ ***A1***

$= \frac{1}{a}$ ***A1***

[3 marks]

b. as $\left\{\frac{1}{n^2}\right\}$ is a positive decreasing sequence we consider the function $\frac{1}{x^2}$

we look at $\int_1^{\infty} \frac{1}{x^2}\mathrm{d}x$ ***M1***

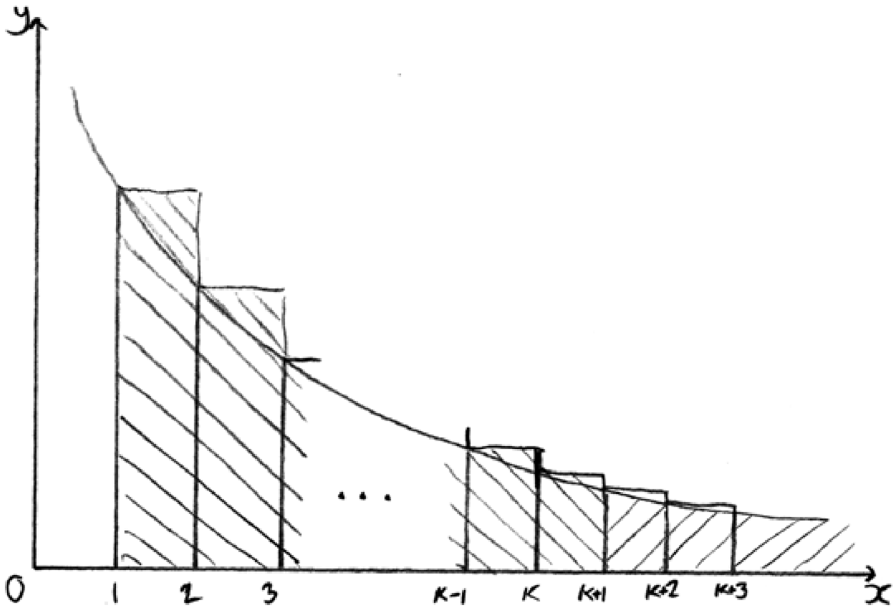
$\int_1^{\infty} \frac{1}{x^2}\mathrm{d}x = 1$ ***A1***

since this is finite (allow “limit exists” or equivalent statement) ***R1***

$\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges ***AG***

[3 marks]

c. (i)



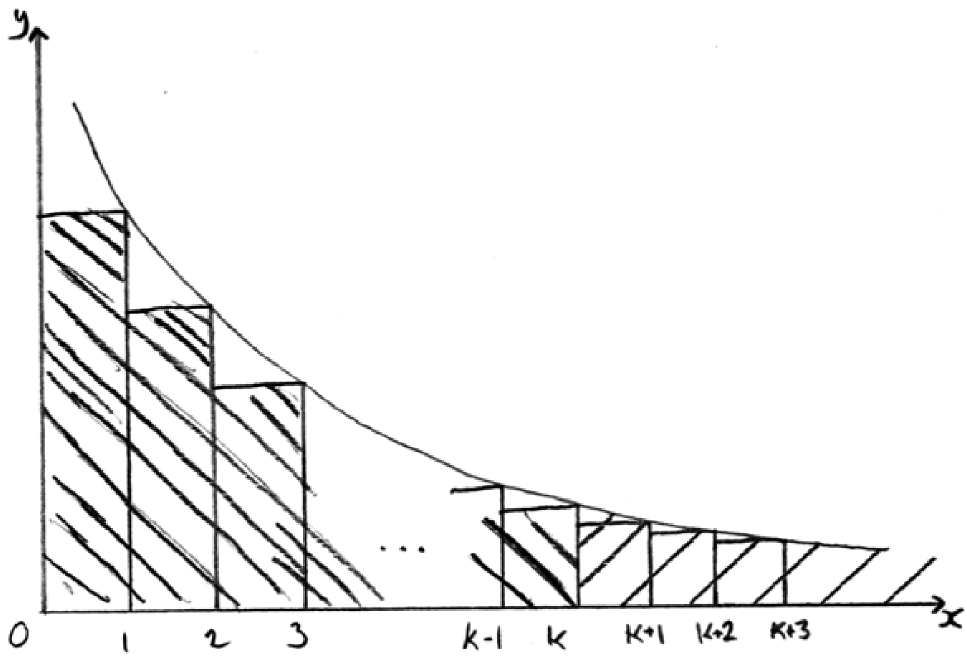
attempt to shade rectangles ***M1***

correct start and finish points for rectangles ***A1***

since the area shaded is less that the area of the required staircase we have ***R1***

$\sum_{n=1}^{k} \frac{1}{n^2} + \int_{k+1}^{\infty} \frac{1}{x^2} \mathrm{d}x < L$ ***AG***

(ii)



attempt to shade rectangles ***M1***

correct start and finish points for rectangles ***A1***

since the area shaded is greater that the area of the required staircase we have ***R1***

$L < \sum_{n=1}^{k} \frac{1}{n^2} + \int_{k}^{\infty} \frac{1}{x^2} \mathrm{d}x$ ***AG***

Note: Alternative shading and rearranging of the inequality is acceptable.

[6 marks]

d. $\int_{k+1}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{k+1}$, $\int_{k}^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{k}$ ***A1A1***

$\sum_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k+1} < L < \sum_{n=1}^{k} \frac{1}{n^2} + \frac{1}{k}$ ***AG***

[2 marks]

e. $\frac{205}{144} + \frac{1}{5} < \frac{\pi^2}{6} < \frac{205}{144} + \frac{1}{4}$ $\left(1.6236... < \frac{\pi^2}{6} < 1.6736...\right)$ ***A1***

$\sqrt{6\left(\frac{205}{144} + \frac{1}{5}\right)} < \pi < \sqrt{6\left(\frac{205}{144} + \frac{1}{4}\right)}$ ***(M1)***

$3.12 < \pi < 3.17$ ***A1 N2***

[3 marks]

Examiners report

a. Most candidates correctly obtained the result in part (a). Many then failed to realise that having obtained this result once it could then simply be stated when doing parts (b) and (d)

b. Most candidates correctly obtained the result in part (a). Many then failed to realise that having obtained this result once it could then simply be stated when doing parts (b) and (d)

In part (b) the calculation of the integral as equal to 1 only scored 2 of the 3 marks. The final mark was for stating that 'because the value of the integral is finite (or 'the limit exists' or an equivalent statement) then the series converges. Quite a few candidates left out this phrase.

c. Most candidates correctly obtained the result in part (a). Many then failed to realise that having obtained this result once it could then simply be stated when doing parts (b) and (d)

Candidates found part (c) difficult. Very few drew the correct series of rectangles and some clearly had no idea of what was expected of them.

d. Most candidates correctly obtained the result in part (a). Many then failed to realise that having obtained this result once it could then simply be stated when doing parts (b) and (d)

e. Though part (e) could be done without doing any of the previous parts of the question many students were probably put off by the notation because only a minority attempted it.

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,\ y)$ where $f(x,\ y) = y - 2x$.

a. Sketch, on one diagram, the four isoclines corresponding to $f(x,\ y) = k$ where k takes the values -1, -0.5, 0 and 1. Indicate clearly where each isocline crosses the y axis. [2]

b. A curve, C, passes through the point $(0, 1)$ and satisfies the differential equation above. [3]

Sketch C on your diagram.

c. A curve, C, passes through the point $(0, 1)$ and satisfies the differential equation above. [1]

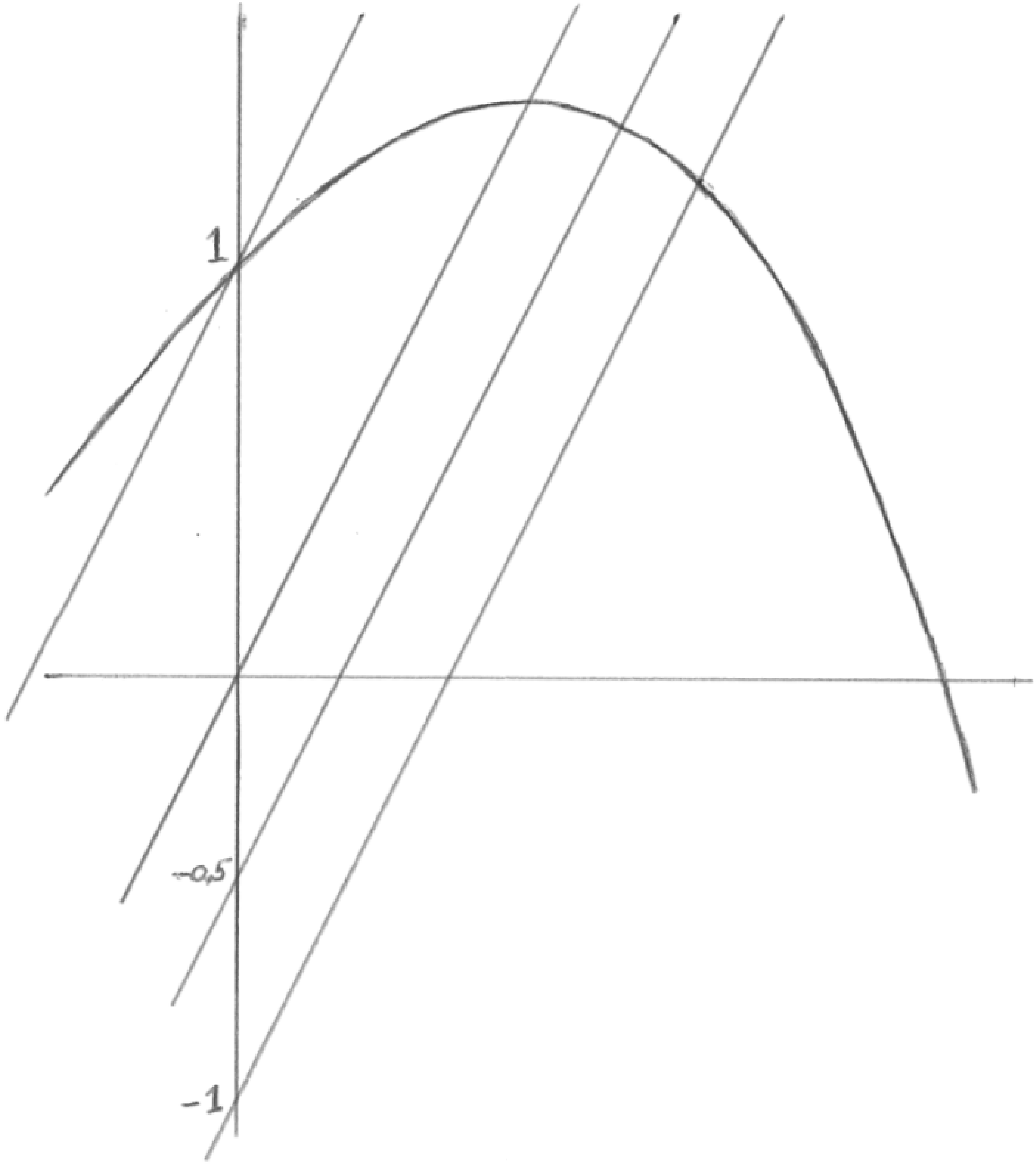
State a particular relationship between the isocline $f(x,\ y) = -0.5$ and the curve C, at their point of intersection.

d. A curve, C, passes through the point $(0, 1)$ and satisfies the differential equation above. [4]

Use Euler's method with a step interval of 0.1 to find an approximate value for y on C, when $x = 0.5$.

Markscheme

a.

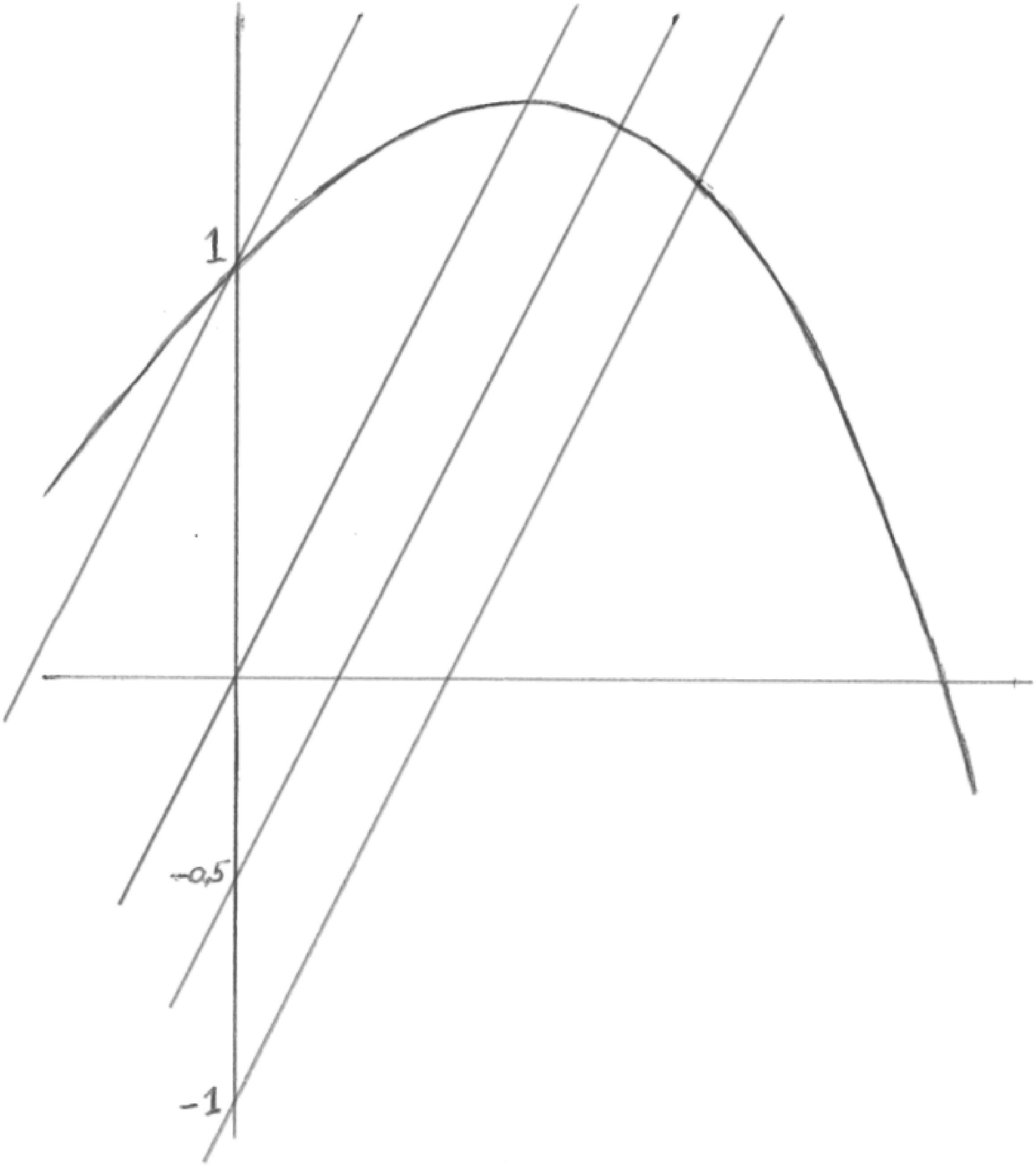


A1 for 4 parallel straight lines with a positive gradient ***A1***

A1 for correct y intercepts ***A1***

[2 marks]

b.



A1 for passing through $(0, 1)$ with positive gradient less than 2

A1 for stationary point on $y = 2x$

A1 for negative gradient on both of the other 2 isoclines ***A1A1A1***

[3 marks]

c. The isocline is perpendicular to C ***R1***

[1 mark]

d. $y_{n+1} = y_n + 0.1(y_n - 2x_n) \quad (= 1.1y_n - 0.2x_n)$ ***(M1)(A1)***

Note: Also award ***M1A1*** if no formula seen but y_2 is correct.

$y_0 = 1,\ y_1 = 1.1,\ y_2 = 1.19,\ y_3 = 1.269,\ y_4 = 1.3359$ ***(M1)***

$y_5 = 1.39$ to 3sf ***A1***

Note: ***M1*** is for repeated use of their formula, with steps of 0.1.

Note: Accept 1.39 or 1.4 only.

[4 marks]

Total [10 marks]

Examiners report

a. Some candidates ignored the instruction to prove from first principles and instead used standard differentiation. Some candidates also only found a derivative from one side.

b. Parts (b) and (c) were attempted by very few candidates. Few recognized that the gradient of the curve had to equal the value of k on the isocline.

c. Parts (b) and (c) were attempted by very few candidates. Few recognized that the gradient of the curve had to equal the value of k on the isocline.

d. Those candidates who knew the method managed to score well on this part. On most calculators a short program can be written in the exam to make Euler's method very quick. Quite a few candidates were losing time by calculating and writing out many intermediate values, rather than just the x and$y$ values.

Let the differential equation $\frac{dy}{dx} = \sqrt{x+y},\ (x+y \geqslant 0)$ satisfying the initial conditions $y = 1$ when $x = 1$. Also let $y = c$ when $x = 2$.

a. Use Euler's method to find an approximation for the value of c , using a step length of $h = 0.1$. Give your answer to four decimal places. [6]

b. You are told that if Euler's method is used with $h = 0.05$ then $c \simeq 2.7921$, if it is used with $h = 0.01$ then $c \simeq 2.8099$ and if it is used with $h = 0.005$ then $c \simeq 2.8121$. [3]

Plot on graph paper, with h on the horizontal axis and the approximation for c on the vertical axis, the four points (one of which you have calculated and three of which have been given). Use a scale of 1 cm = 0.01 on both axes. Take the horizontal axis from 0 to 0.12 and the vertical axis from 2.76 to 2.82.

c. Draw, by eye, the straight line that best fits these four points, using a ruler. [1]

d. Use your graph to give the best possible estimate for c , giving your answer to three decimal places. [2]

Markscheme

a. using $x_0 = 1,\ y_0 = 1$

$x_n = 1 + 0.1n,\ y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$ ***(M1)(M1)(A1)***

Note: If they have not written down formulae but have $x_1 = 1.1$ and $y_1 = 1.14142\ldots$ award ***M1M1A1***.

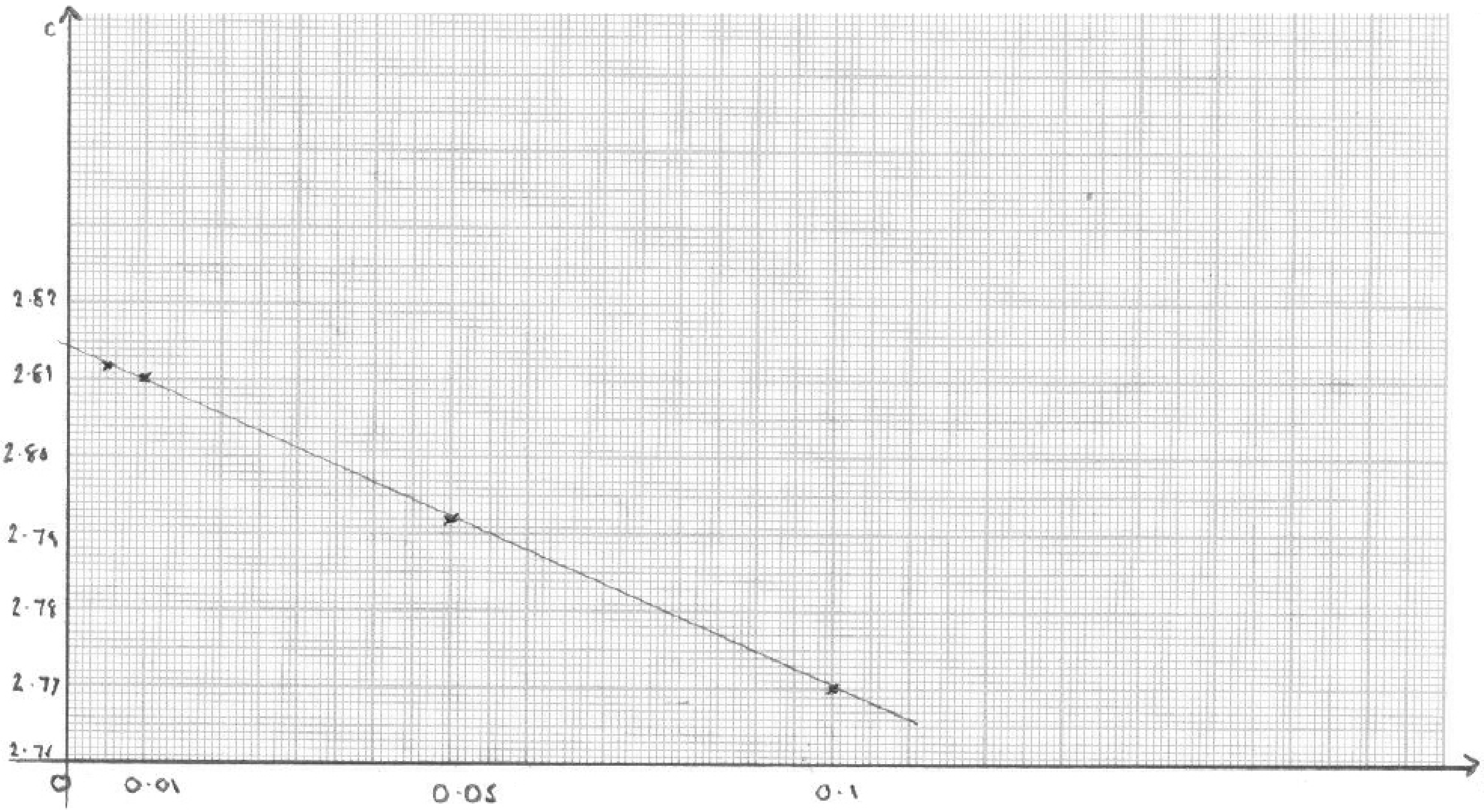
gives by GDC $x_{10} = 2,\ y_{10} = 2.770114792\ldots$ ***(M1)(A1)***

so $a \simeq 2.7701$ (4dp) ***A1*** ***N6***

Note: Do not penalize over-accuracy.

[6 marks]

b.



points drawn on graph above ***A1A1A1***

Note: Award ***A1*** for scales, ***A1*** for 2 points correctly plotted, ***A1*** for other 2 points correctly plotted (second and third ***A1*** dependent on the first being correct).

[3 marks]

c. suitable line of best fit placed on graph ***A1***

[1 mark]

d. letting $\mathrm{h} \to 0$ we approach the *y* intercept on the graph so ***(R1)***

$c \simeq 2.814\ (3\mathrm{dp})$ ***A1***

Note: Accept 2.815.

[2 marks]

Examiners report

a. Part (a) was done well. We would recommend that candidates write down the equation they are using, in this case, $y_{n+1} = y_n + 0.1\sqrt{x_n + y_n}$, to ensure they get all the method marks. Beyond this the answer is all that is needed (or if a student wishes to show working, simply each of the values of x_n and y_n) . Many candidates wasted a lot of time by writing out values of each part of the function, perhaps indicating they did not how to do it more quickly using their calculators.

Part (b) Surprisingly when drawing the graph a lot of candidates had (0.01, 2.8099) closer to 2.80 than 2.81

Most realised that the best possible estimate was given by the *y*-intercept of the line they had drawn.

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